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A Functorial Rectification of Finitely Cocomplete ∞-Categories

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Name	Benni Thien Ngo
E-Mail	xxx
Matrikelnummer	xxx
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Betreuer	Prof. Dr. Denis-Charles Cisinski

Abstract

We prove that the homotopy theory of categories with weak equivalences and cofibrations and the homotopy theory of finitely cocomplete quasi-categories are equivalent. For this, both theories will be presented by categories with weak equivalences and fibrations and their equivalence will follow from the right approximation property of a suitable functor between them.

Contents

Introduction

In classical homotopy theory, one studies properties of topological spaces which hold up to homotopy equivalences, that is, instead of the category of topological spaces, one considers its homotopy category, a category with the same objects, but in which homotopy equivalences become invertible. Motivated by this example, a central notion in homotopy theory is that of a localization[1] of a marked category:

[1] If each category in the definition is allowed to be an ∞-category, the notion of localizations can be adapted verbatim to the setting of ∞-categories.

Given a marked category (C, W), i.e. a category C together with a class W of morphisms in C, a functor $\gamma\colon C \to L_W C$ is called a *localization of C at W* if

- γ is essentially surjective;
- γ sends morphisms in W to isomorphisms in $L_W C$;
- for any category D, the pullback functor

  $$\gamma^*\colon \operatorname{Fun}(L_W C, D) \longrightarrow \operatorname{Fun}(C, D)$$

  is fully faithful and its essential image consists of those functors that send morphisms in W to isomorphisms in D.

In other words, $\gamma\colon C \to L_W C$ universally adds formal inverses for the marked morphism, which will be called weak equivalences in analogy to the homotopy equivalences of the classical homotopy theory, so that $L_W C$ can be thought of as the homotopy theory of C. If one is only concerned about the localization itself, W may be assumed to form a wide subcategory, that is, W contains all identities and is closed under composition, which leads us to the notion of relative categories. [BK12] shows that the homotopy theory of relative categories and that of ∞-categories are equivalent, so that any ∞-category can be presented as the localization of a relative category and one may think of them as homotopy theories. Nevertheless, localizations of arbitrary relative categories are oftentimes unpractical as their morphisms are in general hard to describe. Classically, this led to the notion of model categories, relative categories with two additional classes of auxiliary morphisms, fibrations and cofibrations, and special interrelations between them, but their localizations are bicomplete, which limits the range of homotopy theories that can be presented.

In this thesis, we seek for the middle ground between the arbitrariness of relative categories and the restrictiveness of model categories, namely with categories with weak equivalences and cofibrations. As their name suggests, these are relative categories with only one class of auxiliary morphisms, the cofibrations. By results of chapter 7 in [Cis19] which the name for these objects have been taken from, categories with weak equivalences and cofibrations admit calculi of left fractions, which are useful to control the mapping spaces of their localizations, so that the homotopy theories these categories present are finitely cocomplete. Our main result is to show that, in fact, all finitely cocomplete homotopy theories can be presented by categories with weak equivalences and cofibrations. Originally, this result is due to [Szu14], which establishes an equivalence of the homotopy theory of so-called cofibration categories and of the homotopy theory of finitely cocomplete quasi-categories. Here, the notion of cofibration categories from [Szu14] are more specific than that of categories with weak equivalences and cofibrations, and Section 1 as well as Section 2 will serve as a generalization of the notion of cofibration categories from [Szu14] to quasi-categories with weak equivalences and cofibrations. In Section 3, we will construct a comparison map between the homotopy theory of (quasi-)categories with weak equivalences and cofibrations and the homotopy theory of finitely cocomplete quasi-categories, which we exhibit to be an equivalence in Section 4.

1 Quasi-categories with Weak Equivalences and Cofibrations

The Definition and Basic Properties

Definition 1.1. [Cis19, Definition 7.4.6 & 7.4.12] A *quasi-category with weak equivalences and cofibrations* is a quasi-category C together with an initial object $\varnothing$, a subcategory $wC \subseteq C$ and a wide subcategory $coC \subseteq C$ satisfying the following conditions:

1. Given any commutative triangle in C of the form
$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c, \end{tikzcd}$$
if two out of the three maps f, g and h belong to wC, so does the third.

2. Given any cospan in C of the form
$$b \xleftarrow{i} a \xrightarrow{f} a'$$
such that i and the canonical maps $\varnothing \to a$ and $\varnothing \to a'$ belong to coC, its pushout exists in C such that, for any pushout square in C
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & a' \arrow[d, "i'"] \\ b \arrow[r, "f'"'] & b', \end{tikzcd}$$
i' belongs to coC. Additionally, i' belongs to wC if i does so.

3. Given any map $f\colon a \to b$ in C such that the canonical map $\varnothing \to a$ belongs to coC, there exists a commutative triangle in C of the form
$$\begin{tikzcd} & b' \arrow[dr, "s"] & \\ a \arrow[ur, "i"] \arrow[rr, "f"'] & & b \end{tikzcd}$$
such that i belongs to coC and s belongs to wC.

Maps in wC or coC are called *weak equivalences* or *cofibrations* in C, respectively, and maps that are both weak equivalences and cofibrations are called *trivial cofibrations*. Any object a in C such that the canonical map $\varnothing \to a$ is a cofibration is called *cofibrant*. Under this naming convention, we call the first condition the *2-out-of-3 property for weak equivalences*, the second condition the *pushout axioms for cofibrations and trivial cofibrations* and the third condition the *factorization axiom for maps with cofibrant source.*

Dually, a *quasi-category with weak equivalences and fibrations* is a quasi-category C together with a terminal object $*$, a subcategory $wC \subseteq C$ and a wide subcategory $fiC \subseteq C$ such that $(C^{\mathrm{op}}, *, (wC)^{\mathrm{op}}, (fiC)^{\mathrm{op}})$ is a quasi-category with weak equivalences and cofibrations. Maps in wC are still called weak equivalences whereas maps in fiC are called *fibrations* of which those that are weak equivalences as well are called *trivial fibrations*, and any object a in C such that the canonical map $a \to *$ is a fibration is called *fibrant*.

For the notion of subcategories of quasi-categories, we follow [Lur18, Definition 4.1.2.2], that is, a subcategory of a quasi-category C is a quasi-category C' which is a simplicial subset of C such that the inclusion functor $C' \hookrightarrow C$ is an inner fibration. In other literature, there is also the use of the notion of classes of weak equivalences and cofibrations with additional conditions. The following is a general observation of this duality.

Observation 1.2. Denote the unit of the nerve and realization adjunction by $\eta\colon \mathrm{id}_{\mathrm{sSet}} \to N \circ \mathrm{ho}$, and let C be a quasi-category. Moreover, given any subcategory D of $\mathrm{ho}(C)$, the induced functor $N(D) \to N(\mathrm{ho}(C))$ is an inclusion of a subcategory. Recall that $\eta_C\colon C \to N(\mathrm{ho}(C))$ is an isofibration, so that we may form the pullback square in sSet

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & C \arrow[d, "\eta_C"] \\ N(D) \arrow[r, hook] & N(\mathrm{ho}(C)) \end{tikzcd}$$

and, by [Lur18, Remark 4.1.2.6], the projection $D' \to C$ is an inclusion of a subcategory. By [Lur18, Proposition 4.1.2.10], all subcategories of C are formed in this manner, that is subcategories of C correspond to subcategories of $\mathrm{ho}(C)$, which themselves correspond to classes K of morphisms in $\mathrm{ho}(C)$ that are closed under composition and identities[1].

Therefore, we may as well speak of the classes of weak equivalences and cofibrations in a quasi-category with weak equivalences and cofibrations once we impose these additional conditions:

- For any weak equivalence $a \to b$, the identities of a and b are weak equivalences.
- All identities are cofibrations, and compositions of cofibrations are again cofibrations.

Note that compositions of weak equivalences are already weak equivalences by the 2-out-of-3 property.

Proposition 1.3. *All identities in a quasi-category with weak equivalences and cofibrations are weak equivalences.*

Proof. Let $\varnothing$ be the specified initial object. Then its identity is a cofibration, so that $\varnothing$ is cofibrant. Therefore, given any object a, the canonical map $\varnothing \to a$ can be factorized into a weak equivalence followed by a cofibration. In particular, there exists a weak equivalence of the form $a \to b$, so that the identity of a is a weak equivalence. □

[1]By this, we mean that, for any map $a \to b$ belonging to K, the identities of a and b belong to K.

Remark. This implies that the underlying quasi-category together with the class of weak equivalences forms a *marked simplicial set*, i.e. a simplicial set with a class of edges containing all degenerate edges.

Corollary 1.4. *Sections and retracts of weak equivalences are again weak equivalences.*

Proof. This follows from the preceding proposition and the 2-out-of-3 property for weak equivalences. ☐

Observation 1.5. Any commutative diagram of the form

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b \arrow[r, "g'"'] & b' \end{tikzcd}$$

is a pushout square if f and f' are isomorphisms.

Lemma 1.6. *Any isomorphism with cofibrant source or target is a trivial cofibration. In particular, isomorphisms between cofibrant objects are trivial cofibrations.*

Proof. Let $i\colon a \to b$ be an isomorphism. Without loss of generality, assume that a is cofibrant. Otherwise, proceed the proof with an inverse of i and conclude with the preceding corollary. By the preceding observation,

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "i"] \\ a \arrow[r, "i"'] & b \end{tikzcd}$$

is a pushout square. Per definition, 1_a is a cofibration and, by the preceding proposition, it is a weak eqivalence as well, thus a trivial cofibration. By the pushout axiom, i is a trivial cofibration. ☐

Proposition 1.7. *Let $(C, \varnothing, coC, wC)$ be a quasi-category with weak equivalences and cofibrations. Then $(C, \varnothing', coC, wC)$ is a quasi-category with weak equivalences and cofibrations for any initial object $\varnothing'$ in C.*

Remark. For this reason, we would like to consider initial objects as a property of quasi-categories with weak equivalences and cofibrations rather than as a structure of them, that is, we do not want to specify an initial object, but only require the existence of one.

Proof of proposition. We only need to show that the notion of cofibrant objects does not depend on the choice of the initial object. Let $\varnothing \to a$ belong to coC, i.e. a is cofibrant with respect to $\varnothing$. By initiality, there are canonical isomorphisms $i\colon \varnothing \to \varnothing'$ and $i'\colon \varnothing' \to \varnothing$, which are mutually inverse to each other, so that the canonical map $\varnothing' \to a$ is a composition of i' and the canonical map $\varnothing \to a$. By the previous lemma, i and i' are trivial cofibrations, so that $\varnothing' \to a$ is a cofibration, as a composite of such. Hence, a is cofibrant with respect to $\varnothing'$. ☐

Example 1.8. The notion of quasi-categories with weak equivalences and cofibrations serves as a generalization of the notion of model categories. Therefore, it should not be surprising that (the nerves of) model categories are one immediate example. Another one, that will play an important role, are finitely cocomplete quasi-categories where all maps are declared to be cofibrations and the isomorphisms are the weak equivalences.

The Factorization Axiom and its Implications

We would like to introduce an interesting class of objects in a quasi-category with weak equivalences and cofibrations. For this, we need the following:

Lemma 1.9. *Let C be a quasi-category with weak equivalences and cofibrations. Then finite coproducts of cofibrant objects exist in C and are cofibrant.*

Proof. By induction, it suffices to show that, given two cofibrant objects a and b in C, their coproduct $a \sqcup b$ exists in C and is cofibrant. By the pushout axiom for cofibrations, there is a pushout square in C of the form

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & b \arrow[d, "j"] \\ a \arrow[r, "i"'] & x \end{tikzcd}$$

with i and j cofibrations. By [Cis19, Proposition 6.6.8], x together with i and j is a coproduct of a and b. As the composition of two cofibrations, the canonical map $\varnothing \to x$ is a cofibration, i.e. x is cofibrant. $\square$

Definition 1.10. [Cis19, Definition 2.2.11] A *cylinder object* of a cofibrant object in a quasi-category with weak equivalences and cofibrations is a factorization of its codiagonal into a cofibration followed by a weak equivalence. Dually, a *path object* of a fibrant object in a quasi-category with weak equivalences and fibrations is a factorization of its diagonal into a weak equivalence followed by a fibration.

Example 1.11. This concept is an abstraction of the construction for topological spaces: Given a topological space X, its cylinder $X \times [0,1]$ together with the map $X \sqcup X \to X \times [0,1]$ given by the boundary inclusions and with the collapsing map $X \times [0,1] \to X$ given by the canonical projection forms a cylinder object in both the Quillen-Serre and the Strøm model structure of topological spaces. Cylinders are central in the definition of (left) homotopies and homotopy equivalences and this fact translates well into our abstract setting of quasi-categories C with weak equivalences and cofibrations[1]:

Two maps $f, g\colon a \to b$ between cofibrant objects in C have the same image in $\operatorname{ho} LC$ if there is a cylinder object of a

$$\begin{tikzcd} & \mathrm{cyl}(a) \arrow[dr, "r"] & \\ a \sqcup a \arrow[ur, "\partial"] \arrow[rr, "(1_a,1_a)"'] & & a, \end{tikzcd}$$

[1]An account of the dual case is given in [Cis19, Paragraph 7.5.13 & Proposition 7.6.14].

a trivial cofibration $i\colon b \to b'$ and a map $h\colon \operatorname{cyl}(a) \to b'$ such that

$$\begin{tikzcd} a \sqcup a \arrow[r, "\partial"] \arrow[d, "(f,g)"'] & \operatorname{cyl}(a) \arrow[d, "h"] \\ b \arrow[r, "i"'] & b' \end{tikzcd}$$

is a commutative square in C. In this case, f and g are called *(path-)homotopic* to each other.

Our reason of interest in cylinder objects in this section lies in the fact that they produce almost all factorizations.

Lemma 1.12. *Let C be a quasi-category with initial object and classes of weak equivalences and cofibrations that satisfies all axioms of a quasi-category with weak equivalences and cofibrations, but the factorization axiom, and let $f\colon a \to b$ be a map between cofibrant objects in C. Then f factors into a cofibration followed by a weak equivalence if its source a admits a cylinder object.*

Proof. We need to construct a commutative triangle in C of the form

$$\begin{tikzcd} & b' \arrow[dr, "s"] & \\ a \arrow[ur, "i"] \arrow[rr, "f"'] & & b \end{tikzcd}$$

where i is a cofibration and s is a weak equivalence. By the preceding lemma, there are coproducts of the form

$$\begin{tikzcd} \varnothing_C \arrow[r] \arrow[d] & a \arrow[d, "\iota_2"] \\ a \arrow[r, "\iota_1"'] & a \sqcup a \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \varnothing_C \arrow[r] \arrow[d] & b \arrow[d, "\kappa_2"] \\ a \arrow[r, "\kappa_1"'] & a \sqcup b \end{tikzcd}$$

where the canonical inclusions ι_1, ι_2, κ_1 and κ_2 are cofibrations in C. Let

$$\begin{tikzcd} & \operatorname{cyl}(a) \arrow[dr, "r"] & \\ a \sqcup a \arrow[ur, "\partial"] \arrow[rr, "(1_a,1_a)"'] & & a \end{tikzcd}$$

be a cylinder object of a. Note that the diagram in C

$$\begin{tikzcd} \varnothing_C \arrow[r] \arrow[d] & a \arrow[r, "f"] \arrow[d, "\iota_2"] & b \arrow[d, "\kappa_2"] \\ a \arrow[r, "\iota_1"'] & a \sqcup a \arrow[r, "1_a \sqcup f"'] & a \sqcup b \end{tikzcd} \tag{1.12.1}$$

commutes. Here, $1_a \sqcup f$ is the map induced by the commutative diagram in C

$$\begin{tikzcd} a \arrow[d, "1_a"'] & \varnothing_C \arrow[l] \arrow[r] \arrow[d] & a \arrow[d, "f"] \\ a & \varnothing_C \arrow[l] \arrow[r] & b. \end{tikzcd}$$

In particular, κ_1 is a composition of $1_a \sqcup f$ and ι_1. As the left inner square and the outer rectangle in (1.12.1) are pushout squares, the right inner square is one as well, by the pasting law for pushouts squares. Now, define the so-called *mapping cylinder* $\mathrm{cyl}(f)$ of f by a pushout square in C of the form

$$\begin{tikzcd} a \sqcup a \arrow[r, "1_a \sqcup f"] \arrow[d, "\partial"'] & a \sqcup b \arrow[d, "\delta"] \\ \mathrm{cyl}(a) \arrow[r, "\tilde{f}"'] & \mathrm{cyl}(f). \end{tikzcd}$$

By the pushout axiom, δ is a cofibration, so that the composite of δ and κ_1 is a cofibration as well. We define i to be just this composite. By the pasting law for pushout squares, the outer rectangle in this commutative diagram in C

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\iota_2"'] & b \arrow[d, "\kappa_2"] \\ a \sqcup a \arrow[r, "1_a \sqcup f"] \arrow[d, "\partial"'] & a \sqcup b \arrow[d, "\delta"] \\ \mathrm{cyl}(a) \arrow[r, "\tilde{f}"] & \mathrm{cyl}(f) \end{tikzcd}$$

is a pushout square. Note that $\partial\iota_1$ and $\partial\iota_2$ are cofibrations, as composite of such. Additionally, they are sections of the trivial cofibration r and thus trivial cofibrations by Lemma 1.4. By the pushout axiom, the composite of δ and κ_2 is a trivial cofibration as well. Moreover, one can see that this diagram in C

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\partial\iota_2"'] & b \arrow[d, "1_b"] \\ \mathrm{cyl}(a) \arrow[r, "fr"'] & b \end{tikzcd}$$

commutes. Hence, it induces a map $s\colon\ \mathrm{cyl}(f) \to b$, which is a retract of the trivial cofibration $\delta\kappa_2$ and thus a weak equivalence by Lemma 1.4. Per construction, f is the composite of s and i as we have in formulae:

$$si = s\delta\kappa_1 = s\delta(1_a \sqcup f)\iota_1 = s\tilde{f}\partial\iota_1 = fr\partial\iota_1 = f1_a = f.$$

□

Remark. If we additionally assume that any cofibrant object in C admits a cylinder object, then the factorization axiom holds for any map between cofibrant objects. Therefore, if all objects in C are cofibrant, the existence of cylinder objects for all cofibrant objects is equivalent to the factorization axiom.

Proposition 1.13. *[Cis19, Proposition 7.4.13] Under the same assumptions as in Lemma 1.12, $f\colon a \to b$ can be factored into a cofibration followed by a retract of a trivial cofibration if $a \sqcup b$ admits a cylinder object.*

Proof. By Lemma 1.9, the coproduct $a \sqcup b$ exists and is cofibrant, so that, by Lemma 1.12, the induced map $f \sqcup 1_b\colon a \sqcup b \to b$ factors into a cofibration

$i\colon a \sqcup b \to b'$ followed by a weak equivalence $s\colon b' \to b$. This means that there is a commutative triangle in C of the form

$$\begin{tikzcd} & b' \arrow[dr, "s"] & \\ a \sqcup b \arrow[ur, "i"] \arrow[rr, "f \sqcup 1_b"'] & & b. \end{tikzcd}$$

By precomposition with the canonical inclusion $\kappa_2\colon b \to a \sqcup b$, we get a commutative triangle in C of the form

$$\begin{tikzcd} & b' \arrow[dr, "s"] & \\ b \arrow[ur, "i\kappa_2"] \arrow[rr, "1_b"'] & & b. \end{tikzcd}$$

$i\kappa_2$ is a cofibration, as a composite of such, and, as a section of s, it is a weak equivalence as well. Thus, s is the retract of the trivial cofibration $i\kappa_2$. Analogously, the triangle in C

$$\begin{tikzcd} & b' \arrow[dr, "s"] & \\ a \arrow[ur, "i\kappa_1"] \arrow[rr, "f"'] & & b \end{tikzcd}$$

commutes as well, and, as $i\kappa_1$ is a cofibration, this concludes the proof. $\square$

Remark. In the case that all cofibrant objects in C admit cylinder objects, Lemma 1.13 is known as the *factorization lemma.*

Corollary 1.14 (Ken Brown's lemma)**.** *[Cis19, Corollary 7.4.14] Let C be a quasi-category with weak equivalences and cofibrations, D a quasi-category and V a subcategory of D satisfying the 2-out-of-3 property. Then any functor $F\colon C \to D$ that sends trivial cofibrations between cofibrant objects to maps in V sends weak equivalences between cofibrant to maps in V.*

Proof. Let $F\colon C \to D$ be such a functor and let $w\colon a \to b$ be a weak equivalence between cofibrant objects in C. By the factorization lemma, there are commutative triangles in C

$$\begin{tikzcd} & b' \arrow[dr, "r"] & \\ a \arrow[ur, "i"] \arrow[rr, "w"'] & & b \end{tikzcd}$$

where i is cofibration and

$$\begin{tikzcd} & b' \arrow[dr, "r"] & \\ b \arrow[ur, "j"] \arrow[rr, "1_b"'] & & b \end{tikzcd}$$

where j is trivial cofibration. As an isomorphism between cofibrant objects, 1_b is a trivial cofibration by Lemma 1.6, and r is a weak equivalence by Lemma 1.4.

By the 2-out-of-3 property, i is a weak equivalence and thus a trivial cofibration. By assumption, $F(i)$ is in V and so are $F(j)$ and $F(1_b)$. By the 2-out-of-3 property for V, the map $F(r)$ is in V, so that $F(w)$ is the composition of two maps in V, thus itself in V. $\square$

The Computation of Homotopy Pushouts

The factorization lemma is also useful for various computation, e.g. for that of homotopy pushouts, which is going to be sketched in the following paragraphs.

Observation 1.15. [Cis19, Proposition 7.4.15] Let C be a quasi-category with weak equivalences and cofibrations and let a be a cofibrant object in C. Then the coslice a/C naturally carries the structure of a quasi-category with weak equivalences and cofibrations where weak equivalences and cofibrations are those maps that the canonical projection $a/C \to C$ sends to weak equvialences and cofibrations in C, respectively. As the identity of a is an initial object in a/C such that, for any object (b, f) in a/C, the canonical map from $(a, 1_a) \to (b, f)$ is given by the commutative triangle in C

$$\begin{tikzcd} & a \arrow[dl, "1_a"'] \arrow[dr, "f"] & \\ a \arrow[rr, "f"'] & & b, \end{tikzcd}$$

cofibrant objects in a/C are cofibrations in C with source a. Let $(a/C)_c$ denote the full subcategory of a/C spanned by the cofibrant objects. By the pushout axiom for cofibrations, pushing out along a map $f\colon a \to b$ forms a well-defined functor $f^*\colon (a/C)_c \to (b/C)_c$, and we would like to show that f^* preserves trivial cofibrations. Let the commutative triangle in C

$$\begin{tikzcd} & a \arrow[dl, "i"'] \arrow[dr, "j"] & \\ x \arrow[rr, "u"'] & & y \end{tikzcd}$$

determine a trivial cofibration in $(a/C)_c$, i.e. i and j are cofibrations, and u is a trivial cofibration, and denote its image under f^* by the commutative triangle in C

$$\begin{tikzcd} & b \arrow[dl, "i'"'] \arrow[dr, "j'"] & \\ x' \arrow[rr, "u'"'] & & y'. \end{tikzcd}$$

This means that there are pushout squares in C of the form

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "i'"] \\ x \arrow[r, "f_i"'] & x' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "j"'] & b \arrow[d, "j'"] \\ y \arrow[r, "f_j"'] & y' \end{tikzcd}$$

such that u' is induced by the commutative square in C

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "j'"] \\ x \arrow[r, "f_j u"'] & y'. \end{tikzcd}$$

In particular, there is a commutative diagram in C

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "i'"] \\ x \arrow[r, "f_i"] \arrow[d, "u"'] & x' \arrow[d, "u'"] \\ y \arrow[r, "f_j"] & y' \end{tikzcd}$$

in which the upper inner square and the outer rectangle are pushout squares in C, so that the lower inner square is a pushout square in C as well, by the pasting law for pushouts. Hence, u' is a trivial cofibration in C by the pushout axiom for trivial cofibrations in C. This shows that f^* preserves trivial cofibrations. By Ken Brown's lemma, f^* preserves all weak equivalences. In other words, for any diagram of pushout squares in C

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "i'"] \\ x \arrow[r] \arrow[d, "u"'] & x' \arrow[d, "u'"] \\ y \arrow[r] & y' \end{tikzcd}$$

in which a and b are cofibrant, i is a cofibration and any composition of u and i is a cofibration, u' is a weak equivalence if u is one.

Proposition 1.16. *[Cis19, Proposition 7.4.16] Let C be a quasi-category with weak equivalences and cofibrations. Let $i\colon a \to b$ be a cofibration and $s\colon a \to a'$ a weak equivalence of cofibrant objects in C. Then, for any pushout square in C*

$$\begin{tikzcd} a \arrow[r, "s"] \arrow[d, "i"'] & a' \arrow[d, "i'"] \\ b \arrow[r, "s'"'] & b', \end{tikzcd} \tag{1.16.1}$$

s' is a weak equivalence.

Proof. By the factorization lemma, s factors into a cofibration $j\colon a \to a''$ followed by a retract $r\colon a'' \to a'$ of a trivial cofibration $k\colon a' \to a''$. By the 2-out-of-3 property, r and j are weak equivalences. By the pasting law for pushouts, any pushout square of the form (1.16.1) is the outer rectangle in a diagram of the form

$$\begin{tikzcd} a \arrow[r, "j"] \arrow[d, "i"] & a'' \arrow[r, "r"] \arrow[d, "i''"] & a' \arrow[d, "i'"] \\ b \arrow[r, "j'"'] & b'' \arrow[r, "r'"'] & b' \end{tikzcd} \tag{1.16.2}$$

in which both inner squares are pushout squares. By the pushout axiom for trivial cofibrations, j' is a trivial cofibration, so that, by the 2-out-3 property, it suffices to show that r' is a weak equivalence. Let $\ell\colon a' \to b''$ be a composition of i'' and k, which is again a cofibration. By the pushout axiom for (trivial) cofibrations, there is a pushout square in C of the form

$$\begin{tikzcd} a' \arrow[r, "k"] \arrow[d, "\ell"'] & a'' \arrow[d, "\ell'"] \\ b'' \arrow[r, "k'"'] & b''' \end{tikzcd}$$

with k' a trivial cofibration and ℓ' a cofibration. By choice of ℓ, this diagram in C

$$\begin{tikzcd} a' \arrow[r, "k"] \arrow[d, "\ell"'] & a'' \arrow[d, "i''"] \\ b'' \arrow[r, "1_{b''}"'] & b'' \end{tikzcd}$$

commutes and, thus, corresponds to a map $t\colon b''' \to b''$ such that $1_{b'''} = tk'$ and $i'' = t\ell'$. By Lemma 1.4, t is a weak equivalence. Since r is retract of k, this diagram in C

$$\begin{tikzcd} a' \arrow[r, "k"] \arrow[d, "\ell"'] & a'' \arrow[d, "\ell r"] \\ b'' \arrow[r, "1_{b''}"'] & b'' \end{tikzcd}$$

commutes and corresponds to a map $r'''\colon b''' \to b''$ such that $1_{b'''} = r'''k'$ and $\ell r = r'''\ell'$. By Lemma 1.4, r''' is a weak equivalence. Moreover, the horizontal maps in the following commutative diagram

$$\begin{tikzcd} a' \arrow[r, "k"] \arrow[d, "\ell"] & a'' \arrow[r, "r"] \arrow[d, "\ell'"] & a' \arrow[d, "\ell"] \\ b'' \arrow[r, "k'"'] & b''' \arrow[r, "r'''"'] & b'' \end{tikzcd}$$

are identities, so that the outer rectangle is trivially a pushout square. As the left inner square is defined to be a pushout square, so is the right inner square by the pasting law for pushouts. By the same law, the right inner square in (1.16.2) decomposes into two pushout squares

$$\begin{tikzcd} a'' \arrow[r, "r"] \arrow[d, "\ell'"'] & a' \arrow[d, "\ell"] \\ b''' \arrow[r, "r'''"] \arrow[d, "t"'] & b'' \arrow[d, "t'"] \\ b'' \arrow[r, "r'"] & b' \end{tikzcd}$$

where t' is the map induced by the commutative square in C

$$\begin{tikzcd} a'' \arrow[r, "r"] \arrow[d, "\ell'"'] & a' \arrow[d, "i'"] \\ b''' \arrow[r, "r't"'] & b'. \end{tikzcd}$$

Recall that this square commutes because of $r'i'' = i'r$ and $i'' = t\ell'$. By the preceding observation, t' is a weak equivalences, so that r' is a weak equvialences by the 2-out-of-3 property. □

By the pushout axiom for cofibrations, this proposition shows that the class of cofibrations in the subcategory of cofibrant objects forms an example of the following notion.

Definition 1.17. [Cis19, Definition 7.2.22] Let C be a quasi-category with a subcategory W. Then a *W-cofibration*[1] in C is a map $f\colon a \to b$ such that, for any diagram in C

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & a' \arrow[r, "u"] \arrow[d] & a'' \arrow[d] \\ b \arrow[r] & b' \arrow[r, "u'"] & b'' \end{tikzcd}$$

in which the inner squares are pushouts squares, u' belongs to W if u does so. Dually, *W-fibrations* in C are W^{op}-cofibrations in C^{op}.

The reason of interest in W-cofibrations lies in the fact that they compute homotopy pushouts. To be able to formulate this fact, we need the following notions

Definition 1.18. [Cis19, Definition 7.2.20] A *subcategory of acyclic cofibrations* with respect to W is a subcategory W' of W which is closed under compositions and pushouts such that any map in W factors into a map in W' followed by a retract of a map in W'. Dually, a *subcategory of acyclic fibrations* with respect to W is a subcategory W' of W such that $(W')^{\mathrm{op}}$ is a subcategory of trivial cofibrations with respect to W^{op}.

Remark. In [Cis19], these are called subcategories of trivial (co-)fibrations and the reason that we choose to use the attribute "acyclic" instead is to avoid ambiguity because, in general, the trivial (co-)fibrations of a quasi-category with weak equivalences and (co-)fibrations do not form a subcategory of acyclic (co-)fibrations. By the factorization lemma, this is the case when all objects are cofibrant.

Definition 1.19. Let C be a quasi-category with a subcategory W and denote its localization functor by $\gamma\colon C \to L_W C$. Then a commutative square in C is called *homotopy pushout square* if γ sends it to a pushout square in $L_W C$.

Theorem 1.20. *[Cis19, Theorem 7.2.25] Let C be a quasi-category with a subcategory W that satisfies the 2-out-of-3 property and admits a subcategory of*

[1] There are various names for this notion. See [nLa24, 5. References].

acyclic cofibrations. Then any pushout square in C of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & a' \arrow[d] \\ b \arrow[r] & b' \end{tikzcd}$$

with f a W-cofibration is a homotopy pushout square.

Remark. Given a quasi-category C with weak equivalences and cofibrations, this theorem a priori only applies to the full subcategory C_c of cofibrant objects. Hence, cofibrations between cofibrant objects compute homotopy pushouts.

Conversely, any map that computes homotopy pushouts is a W-cofibration.

Proposition 1.21. *Let C be a quasi-category with a subcategory W that is saturated*[1] *and admits a subcategory of acyclic cofibrations. Let $f\colon a \to b$ be a map in C such that any pushout square in C of the form*

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & a' \arrow[d] \\ b \arrow[r] & b' \end{tikzcd}$$

is a homotopy pushout. Then f is a W-cofibration.

Proof. Consider a commutative diagram in C of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "(1)"] & a' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "(2)"] & a'' \arrow[d] \\ b \arrow[r] & b' \arrow[r, "u'"] & b'' \end{tikzcd}$$

where the inner squares (1) and (2) are pushout squares. By the pasting law for pushouts in C, the outer rectangle, which will be denoted by (1+2), is a pushout square as well. By assumption, (1) and (1+2) are homotopy pushout squares. Again by the pasting law for pushouts in $L_W C$, (2) is a homotopy pushout square. Hence, if u is a weak equivalence, then its image in $L_W C$ is an isomorphism and so is the image of u' in $L_W C$. Thus, due to saturation, u' itself is a weak equivalence in C. $\square$

1.22. Let C be a quasi-category with subcategory W and denote the localization functor by $\gamma\colon C \to L_W C$. If W is closed under (compositions and) pushouts, then there is a calculus of left fractions[2] of W in C by [Cis19, Theorem 7.2.16]. By [Cis19, Remark 7.2.10], this implies that any map $a \to b$ in $\operatorname{ho} L_W C$ is of the form $\gamma(s)^{-1}\gamma(f)$ where $f\colon a \to b'$ is a map and $s\colon b \to b'$ is a weak equivalence in C. Additionally, any two such spans in C

$$a \xrightarrow{f} b' \xleftarrow{s} b \quad \text{and} \quad a \xrightarrow{g} b'' \xleftarrow{t} b$$

[1]This means that any map in C belongs to W if and only if its image in $L_W C$ is an isomorphism.

[2]See [Cis19, Definition 7.2.6].

for which there is a weak equivalence $r\colon b' \to b''$ in C such that

$$\begin{tikzcd} & b' \arrow[d, "r"] & \\ a \arrow[ur, "f"] \arrow[dr, "g"'] & & b \arrow[ul, "s"'] \arrow[dl, "t"] \\ & b'' & \end{tikzcd}$$

is a commutative diagram in C represent the same map $a \to b$ in $\operatorname{ho} L_W C$.

Remark. In the presence of a calculus of fractions, the 2-out-of-6 property and the property of being saturated are equivalent for the class of weak equivalences. Recall that the former property states that, given a triple (f, g, h) of composable maps such that $g \circ f$ and $h \circ g$ are weak equivalences, all maps f, g and h are weak equivalences. The latter property states that any map that the localization functor sends to an isomorphism is a weak equivalences. Clearly, being saturated implies the 2-out-of-6 property as the class of isomorphisms satisfies the latter. The converse is a simple computation for which we would like to refer to [KS06, Proposition 7.1.20].

Although the class of weak equivalences in a quasi-category with weak equivalences and cofibrations is generally not saturated, using the factorization axiom and its implications, one can still show that all pushouts in the localization come from homotopy pushouts, which themselves come from pushouts of cofibrations. Note that this does not imply that all maps that compute homotopy pushouts are cofibrations, but weakly equivalent to such.

1.23. Let C be a quasi-category with weak equivalences and cofibrations and denote its full subcategory of cofibrant objects by C_c. Then the class of trivial cofibrations admits a calculus of left fractions in C_c. By Ken Brown's lemma, the localization of C_c at trivial cofibrations is equivalent to the localization of C_c at weak equivalences. Together with the factorization lemma, this implies that any map in $\operatorname{ho}(LC_c)$ is of the form $\gamma(s)^{-1}\gamma(f)$ with f a cofibration and s a trivial cofibration. Therefore, any span in LC_c is isomorphic[1] to (the image of) a span in C_c where at least one leg is a cofibration. By the pushout axiom for cofibrations, the pushout exists and, by the preceding theorem, γ preserves these pushouts. Hence, any pushout in LC_c is the image under γ of a pushout of cofibrations in C_c.

The following fact allows us to use these computations also in arbitrary quasi-categories with weak equivalences and cofibrations.

Theorem 1.24. *[Cis19, Theorem 7.5.18] Let C be a quasi-category with weak equivalences and cofibrations and denote the inclusion of its full subcategory of cofibrant objects by $i\colon C_c \hookrightarrow C$. Then the total (left) derived functor[2] $\bar{\imath}\colon LC_c \to LC$ of i is an equivalence of ∞-categories.*

To conclude this computation, we would like to summarize the fact that cofibrations between cofibrant objects compute all pushouts in the localization.

[1] These isomorphisms come from trivial cofibrations in C_c.
[2] See Lemma 2.2 for a definition and Lemma 3.10 for a specific construction.

Corollary 1.25. *Let C be a quasi-category with weak equivalences and cofibrations. Then any pushout square in C of the form*

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & a' \arrow[d] \\ b \arrow[r] & b' \end{tikzcd}$$

in which a and a' are cofibrant and f is a cofibration is a homotopy pushout square. Conversely, all pushout squares in LC are images of pushout squares in C of the above form, i.e. these homotopy pushouts compute all pushouts in LC. In particular, LC has all pushouts.

Right Exact Functors

Since localization functors are both final and cofinal[1], the localization LC admits initial objects and $\gamma \colon C \to LC$ preserves them. Therefore, LC is a finitely cocomplete ∞-category and, if presented by a quasi-categories, it carries a natural structure of a quasi-category with weak equivalences and cofibrations, so that the localization functor $\gamma \colon C \to LC$ is the first non-trivial instance of the following notion.

Definition 1.26. [Cis19, Definition 7.5.2] Let C and D be quasi-categories with weak equivalences and cofibrations. Then a *right exact functor* from C to D is a functor $F \colon C \to D$ between the underlying quasi-categories satisfying the following conditions:

- For any initial object $\varnothing_C$ in C, the object $F(\varnothing_C)$ is initial in D.
- For any cofibrant object a and any cofibration $p \colon a \to b$ in C, the map $F(p) \colon F(a) \to F(b)$ is a cofibration in D.
- For any cofibrant object a and any trivial cofibration $p \colon a \to b$ in C, the map $F(p) \colon F(a) \to F(b)$ is a trivial cofibration in D.
- For any pushout square in C

  $$\begin{tikzcd} a \arrow[r] \arrow[d, "p"] & a' \arrow[d] \\ b \arrow[r] & b', \end{tikzcd}$$

  such that a and a' are cofibrant and p is a cofibration, the induced diagram in D

  $$\begin{tikzcd} F(a) \arrow[r] \arrow[d, "F(p)"] & F(a') \arrow[d] \\ F(b) \arrow[r] & F(b') \end{tikzcd}$$

  is pushout square.

[1] See [Cis19, Proposition 7.1.10]

In other words, a functor between quasi-categories with weak equivalences and cofibrations is right exact if it preserves

- initial objects,
- cofibrations with cofibrant source,
- trivial cofibrations with cofibrant source and
- pushouts of cofibrations with cofibrant source along any map with cofibrant target.

Dually, *left exact functors* between quasi-categories with weak equivalences and fibrations are those functors $F\colon C \to D$ for which $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ are right exact.

Remark. Ken Brown's lemma implies that right exact functors preserve weak equivalences between cofibrant objects.

Proposition 1.27. *Let C be a quasi-category with weak equivalences and cofibrations and let $F\colon C' \to C$ be a weak categorical equivalence. Then there is a structure of a quasi-category with weak equivalences and cofibrations on C' such that F is right exact.*

Proof. The subcategories coC' and wC' of C' are defined by pullback squares of the form

$$\begin{tikzcd} coC' \arrow[r, hook] \arrow[d] & C' \arrow[d, "F"] \\ coC \arrow[r, hook] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} wC' \arrow[r, hook] \arrow[d] & C' \arrow[d, "F"] \\ wC \arrow[r, hook] & C. \end{tikzcd}$$

This means that a weak equivalence or a cofibration in C' is a map that F sends to such in C. Therefore, one can easily check that coC' is a wide subcategory, i.e. all identities are cofibrations, and that wC' satisfies the 2-out-of-3 property. Since weak categorical equivalences of quasi-categories present equivalences of ∞-categories[1], F preserves and reflects all colimits. In particular, C' admits an initial object $\varnothing_{C'}$ such that $F(\varnothing_{C'})$ is initial in C. Moreover, C' satisfies the pushout axioms for cofibrations and trivial cofibrations, and it remains to show the factorization axiom. Let $f\colon x \to y$ be a map in C' with cofibrant source. By the factorization axiom in C, there is a commutative triangle in C

$$\begin{tikzcd} & b \arrow[dr, "t"] & \\ F(x) \arrow[ur, "j"] \arrow[rr, "F(f)"] & & F(y) \end{tikzcd} \tag{1.27.1}$$

where j is a cofibration and t is a weak equivalence. Because F is essentially surjective and because isomorphisms between cofibrant objects are trivial cofibrations, we may assume that b is of the form $F(y')$ for some object y' in C'. As F is fully faithful, there is a commutative diagram in C

$$\begin{tikzcd} & y' \arrow[dr, "s"] & \\ x \arrow[ur, "i"] \arrow[rr, "f"] & & y \end{tikzcd}$$

[1] See [Cis19, Corollary 3.6.6].

that F sends to the diagram in (1.27.1). Thus, i is a cofibration and s is a weak equivalence in C'. This shows that C' is a quasi-category with weak equivalences and cofibrations and F is right exact per construction. $\square$

Remark. If we denote the quasi-inverse of $F\colon C' \to C$ by $G\colon C \to C'$, then G is right exact with respect to the above structure on C'. To see this, let $f\colon a \to b$ be a map in C. Since the composite $FG\colon C \to C$ is again a weak categorical equivalence between quasi-categories, it is, in particular, fully faithful, so that there is a commutative square in C of the form

$$\begin{tikzcd} (FG)(a) \arrow[r, "(FG)(f)"] \arrow[d, "i"'] & (FG)(b) \\ a \arrow[r, "f"'] & b \arrow[u, "j"'] \end{tikzcd}$$

where i and j are isomorphisms. If f is a weak equivalence or a cofibration between cofibrant objects, then i and j are trivial cofibrations by Lemma 1.6, so that $(FG)(f)$ is a weak equivalence or, respectively, a cofibration, as a composite of such. Per construction of wC' and coC', this implies that $G(f)$ is a weak equivalence or a cofibration in C', respectively. As G is a weak categorical equivalence between quasi-categories itself, it preserves all colimits, in particular the ones of interest. Hence, G is right exact.

Therefore, the above defined structure on C' is unique in the sense that, for any structure $(C', \mathrm{Co}, \mathrm{We})$ of a quasi-category with weak equivalences and cofibrations such that both F and G are right exact, there are canonical right exact functors between (C', coC', wC') and $(C', \mathrm{Co}, \mathrm{We})$ whose underlying functors between quasi-categories are mutually inverse weak categorical equivalences, namely the composite functors $FG\colon (C', coC', wC') \to (C', \mathrm{Co}, \mathrm{We})$ and $FG\colon (C', \mathrm{Co}, \mathrm{We}) \to (C', coC', wC')$.

Observation 1.28. The preceding proposition allows us to disregard any specific choices of initial objects in the notion of right exact functors in the following sense: For any right exact functor $F\colon C \to D$ and any specific choice of initial objects $\varnothing_C$ in C and $\varnothing_D$ in D, there is, up to weak categorical equivalence, a unique right exact functor $F'\colon C \to D$ such that $F'(\varnothing_C) = \varnothing_D$. Denote the structure on D by $(D, \varnothing_D, coD, wD)$. By assumption, $F(\varnothing_C)$ is initial in D, so that the pair of functors $D \to D$ that swap $F(\varnothing_C)$ and $\varnothing_D$ are mutually inverse weak categorical equivalences. By Lemma 1.7, $(D, F(\varnothing_C), coD, wD)$ is a quasi-category with weak equivalences and cofibrations, and one can immediately see that these weak categorical equivalences form right exact functors between $(D, F(\varnothing_C), coD, wD)$ and $(D, \varnothing_D, coD, wD)$. Hence, we can construct the right exact functor $F'\colon C \to D$ by composing $F\colon C \to D$ with the functor $D \to D$ that sends $F(\varnothing_C)$ to $\varnothing_D$.

For this reason, we have not required right exact functors to preserve the specific choices of initial objects, and, with this, we may conclude the remarks of Lemma 1.7.

From now on, we will fix a Grothendieck universe $\mathbf{U}$ and denote the ∞-category of $\mathbf{U}$-small spaces by $\mathcal{S}$. Moreover, given an ∞-category X, its Yoneda embedding will be denoted by よ$_X\colon X \to \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S})$.[1]

[1] よ is the first letter of the name "Yoneda" in Hiragana.

Definition 1.29. Let CofQCat be the category whose objects are $\mathbf{U}$-small quasi-categories with weak equivalences and cofibrations and whose morphisms are right exact functors where the composition law is given by the usual composition of functors between quasi-categories. Dually, let FibQCat be the category of $\mathbf{U}$-small quasi-categories with weak equivalences and fibrations and left exact functors. Note that dualization forms an isomorphism of the categories CofQCat and FibQCat.

2 The Homotopy Theory of Quasi-categories with Weak Equivalences and Cofibrations

If quasi-categories with weak equivalences and cofibrations are considered to be presentations of homotopy theories, a natural question should be when or whether presentations belong to the same homotopy theory. Formally, this means that, given two quasi-categories C and D with weak equivalences and cofibrations, the corresponding localizations LC and LD are equivalent ∞-categories. It will turn out that this equivalence of ∞-categories itself is presented by a zig-zag of right exact functors, but, in this section, we would like to start by studying right exact functors $C \to D$ that present equivalences of ∞-categories $LC \to LD$. That is, we study the homotopy theory of CofQCat arising from the identification of quasi-categories with weak equivalences and cofibrations that present equivalent homotopy theories.

Definitions and Examples

First, we need to formalize what it means for a right exact functor $F\colon C \to D$ to "present" a functor $LC \to LD$. If we denote the localization functors by $\gamma_C\colon C \to LC$ and $\gamma_D\colon D \to LD$, respectively, a common approach is to consider filling problems of the form

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, dashed] & LD. \end{tikzcd}$$

Oftentimes, there is no functor $LC \to LD$ such that the resulting square commutes in any appropriate sense, but one may still ask for a best approximation to such a functor, which leads us to the notion of *Kan extensions*.

Definition 2.1. Let $\mathcal{F}\colon X \to Y$ be a functor of $\mathbf{U}$-small ∞-categories and Z a $\mathbf{U}$-small ∞-category. Then the functor

$$\mathcal{F}^*\colon \operatorname{Fun}(Y,Z) \longrightarrow \operatorname{Fun}(X,Z)$$

defined by precomposition with $\mathcal{F}$ is called *pullback functor*. Its left adjoint

$$\mathcal{F}_!\colon \operatorname{Fun}(X,Z) \longrightarrow \operatorname{Fun}(Y,Z)$$

is called the *left Kan extension* operation along $\mathcal{F}$ whereas the right adjoint of $\mathcal{F}^*$

$$\mathcal{F}_*\colon \operatorname{Fun}(X,Z) \longrightarrow \operatorname{Fun}(Y,Z)$$

is the *right Kan extension* operation along $\mathcal{F}$.

Remark. In the case that Z is the ∞-category $\mathcal{S}$ of spaces, these notions exist by [Cis19, Proposition 6.1.14 & Corollary 6.3.10], but in general this is not the case. Following [Cis19, Paragraph 7.5.23 & Definition 7.5.26], we will use the notion of local Kan extensions to define total derived functors, i.e. to ask that

$$\operatorname{Hom}_{\operatorname{Fun}(X,Z)}\left(\mathcal{F}^*(-),F\right)\colon \operatorname{Fun}(Y,Z)^{\mathrm{op}} \longrightarrow \mathcal{S}$$

is representable or, respectively, that

$$\mathrm{Hom}_{\mathrm{Fun}(X,Z)}\left(F, \mathcal{F}^*(-)\right) : \mathrm{Fun}(Y,Z) \longrightarrow \mathcal{S}$$

is co-representable, for a functor $F\colon Y \to Z$.

Definition 2.2. Let $F\colon C \to D$ be a right exact functor between $\mathbf{U}$-small quasi-categories with weak equivalences and cofibrations and let $\gamma_C\colon C \to LC$ and $\gamma_D\colon D \to LD$ denote the localization functors. Then a *total left derived functor* of F is a representative of the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*(-), \gamma_D F\right) : \mathrm{Fun}(LC,LD)^{\mathrm{op}} \longrightarrow \mathcal{S}$$

and denoted by $\mathbf{L}F\colon LC \to LD$. Dually, a *total right derived functor* of a left exact functor $F\colon C \to D$ is a corepresentative of the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left(\gamma_D F, (\gamma_C)^*(-)\right) : \mathrm{Fun}(LC,LD) \longrightarrow \mathcal{S}$$

and denoted by $\mathbf{R}F\colon LC \to LD$.

Remark. In the case that the left and right Kan extensions along γ_C exist, $\mathbf{L}F$ is the right Kan extension of $\gamma_D F$ along γ_C and $\mathbf{R}F$ is the left Kan extension of $\gamma_D F$ along γ_C, i.e. $\mathbf{L}F = (\gamma_C)_*(\gamma_D F)$ and $\mathbf{R}F = (\gamma_C)_!(\gamma_D F)$.

More importantly, it should be noted that this notion can be defined in a more general setting, e.g. when $F\colon C \to D$ is just a morphisms of simplicial sets and C and D are marked simplicial sets.

Proposition 2.3. *[Cis19, Lemma 7.5.24] Let C and D be $\mathbf{U}$-small ∞-categories with weak equivalences*[1] *and denote the localization functors by $\gamma_C\colon C \to LC$ and $\gamma_D\colon D \to LD$, respectively. Let $F\colon C \to D$ be a functor such that the composite $\gamma_D F\colon C \to LD$ sends all weak equivalences to isomorphisms. Then the functor $\overline{\gamma_D F}\colon LC \to LD$ induced by the universal property of LC, is both a total left and a total right derived functor of F.*

Proof. Recall that $\overline{\gamma_D F}$ is a functor for which $\overline{\gamma_D F}\gamma_C$ is isomorphic to $\gamma_D F$ in $\mathrm{Fun}(C, LD)$. Let $\epsilon\colon \overline{\gamma_D F}\gamma_C \to \gamma_D F$ be a such an isomorphism and let $(\gamma_C)^* : \mathrm{Fun}(LC,LD) \to \mathrm{Fun}(C,LD)$ denote the pullback functor. Then we have $\overline{\gamma_D F}\gamma_C = (\gamma_C)^*\left(\overline{\gamma_D F}\right)$ and composing with ϵ forms the isomorphisms in $\mathrm{Fun}(\mathrm{Fun}(LC,LD),\mathcal{S})$:

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*(-), (\gamma_C)^*\left(\overline{\gamma_D F}\right)\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*(-), \gamma_D F\right)$$
$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left(\gamma_D F, (\gamma_C)^*(-)\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*\left(\overline{\gamma_D F}\right), (\gamma_C)^*(-)\right).$$

By the definition of localization functors, $(\gamma_C)^*$ is fully faithful, so that there are isomorphisms in $\mathrm{Fun}(\mathrm{Fun}(LC,LD),\mathcal{S})$ of the form:

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(LC,LD)}\left(-, \overline{\gamma_D F}\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*(-), (\gamma_C)^*\left(\overline{\gamma_D F}\right)\right)$$
$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(LC,LD)}\left(\overline{\gamma_D F}, -\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*\left(\overline{\gamma_D F}\right), (\gamma_C)^*(-)\right).$$

This shows that $\overline{\gamma_D F}$ represents $\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left((\gamma_C)^*(-), \gamma_D F\right)$ and corepresents $\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,LD)}\left(\gamma_D F, (\gamma_C)^*(-)\right)$, which concludes the proof. □

[1] This means that there are subcategories whose morphisms we call weak equivalences.

Remark. In this case, the total left and total right derived functors of F coincide and, for this reason, we might speak of the *total derived functor* without mentioning the sidedness. Note that the conditions of this lemma are met by functors between marked simplicial sets that preserve the markings, e.g. by right exact functors whose domain consists of only cofibrant objects.

Corollary 2.4. *[Cis06, Proposition 7.5.6] Let $F\colon C \to D$ be a right exact functor between quasi-categories with weak equivalences and cofibrations and assume that all objects in C are cofibrant. Then the total left derived functor $\mathbf{L}F\colon LC \to LD$ is right exact.*

Remark. By Lemma 1.25, LC and LD are finitely cocomplete, so that, if presented by quasi-categories, they carry a natural structure of quasi-categories with weak equivalences and cofibrations where the former are the isomorphisms and the latter are all maps.

Proof of corollary. We need to show that $\mathbf{L}F$ preserves finite limits, i.e. initial objects and pushouts. By the preceding proposition, $(\gamma_C)^*(\mathbf{L}F)$ is naturally isomorphic to $\gamma_D F$. Since F and localization functors preserve initial objects, so does $\mathbf{L}F$. By Lemma 1.25, pushouts in LC and LD are computed by pushouts of cofibrations in C and D, respectively, which F preserves by Lemma 1.25. Therefore, $\mathbf{L}F$ preserves pushouts. □

Definition 2.5. [Szu14, Definition 1.7] A *weak equivalence* in CofQCat is a right exact functor between quasi-categories with weak equivalences and cofibrations whose total left derived functor is an equivalence of ∞-categories. Informally, this means weakly equivalent quasi-categories with weak equivalences and cofibrations present the same homotopy theory.

Example 2.6. For obvious reasons, identities of quasi-categories with weak equivalences and cofibrations are weak equivalences as their total derived functors are again identities.

Example 2.7. Let C be a quasi-category with weak equivalences and cofibrations. Its full subcategory $C_c \subseteq C$ of cofibrant objects forms a quasi-category with weak equivalences and cofibrations where the weak equivalences and cofibrations are those with cofibrant source and target. By Lemma 1.24 citing [Cis19, Theorem 7.5.18], the inclusion functor $i\colon C_c \hookrightarrow C$ is a right exact functor, which is a weak equivalence of quasi-categories with weak equivalences and cofibrations. In other words, all the homotopical data of a quasi-category with weak equivalences and cofibrations is entirely contained in its subcategory of cofibrant objects and the importance of this has been outlined in the previous section.

Example 2.8. [Cis19, Remark 7.1.5] Let C be a marked simplicial set with a localization functor $\gamma\colon C \to LC$. Then the *saturation* $\overline{C}$ of C is defined to be the underlying simplicial set of C where all edges that γ sends to isomorphisms in LC are marked. That is, $\gamma^{-1}\left((LC)^{\simeq}\right)$ is the subcategory of marked edges, itself given by the pullback of simplicials sets:

$$\begin{tikzcd} \gamma^{-1}\left((LC)^{\simeq}\right) \arrow[r, hook] \arrow[d] & C \arrow[d, "\gamma_C"] \\ (LC)^{\simeq} \arrow[r, hook] & LC. \end{tikzcd}$$

Furthermore, using the fact that any morphism $C \to D$ of simplicial sets that sends marked edges to isomorphisms factors through LC and that morphisms of simplicial sets preserve isomorphisms, one can check that such a morphism already sends the marked edges of $\overline{C}$ to isomorphisms. Therefore, γ forms a localization functor of $\overline{C}$ as well, so that the identity of C induces a morphism $s\colon C \to \overline{C}$ whose total derived functor is the identity of LC, an equivalence of ∞-categories.

If C is a quasi-category with weak equivalences and cofibrations, then so is $\overline{C}$ where the class of cofibrations remain unchanged. The only non-trivial point to verify is the pushout axiom for trivial cofibrations: let $i\colon a \to b$ be a trivial cofibration in $\overline{C}$ with a cofibrant and let $f\colon a \to a'$ be a map in $\overline{C}$ with a' cofibrant. By Lemma 1.9, the coproduct $a' \sqcup_a b$ exists. Now, given a pushout square in $\overline{C}$ of the form

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & a' \arrow[d, "i'"] \\ b \arrow[r] & b', \end{tikzcd}$$

i' is already a cofibration in $\overline{C}$ and it remains to show that i' is a weak equivalence in $\overline{C}$. By Lemma 1.25, this diagram in C is a homotopy pushout square. Per assumption, $\gamma(i)$ is an isomorphism in LC and, as the class of isomorphisms is closed under pushouts, $\gamma(i')$ is an isomorphisms as well. Therefore, i' is a weak equivalence in $\overline{C}$. Hence, $\overline{C}$ is a quasi-category with weak equivalences and cofibrations, and the canonical functor $s\colon C \to \overline{C}$ is a (right exact) weak equivalence of quasi-categories with weak equivalences and cofibrations.

The notion of total derived functors depends highly on the notion of weak equivalences as demonstrated by the following.

Example 2.9. The identity functor of sSet induces a Quillen adjunction between the Kan-Quillen and the Joyal model structure on simplicial sets, which is not a Quillen equivalence, i.e. this adjunction does not produce an equivalence of the respective localizations. This means that left or right exact functors which are equivalence of the underlying quasi-categories are generally not weak equivalences of quasi-categories with weak equivalences and fibrations or cofibrations, respectively.

Meanwhile, cofibrations are not necessary to define total derived functors, but still useful for computational reasons, e.g. to show that they exist in the case of quasi-categories with weak equivalences and cofibrations.

Proposition 2.10. *[Cis19, Paragraph 7.5.25] Let $F\colon C \to D$ be a right exact functor between quasi-categories with weak equivalences and cofibrations. Denote the inclusion of the full subcategory of cofibrant objects by $i\colon C_c \hookrightarrow C$, the localization functors by $\gamma_C\colon C \to LC$, $\gamma_{C_c}\colon C_c \to LC_c$ and $\gamma_D\colon D \to LD$ and the total derived functor of i and its quasi-inverse by $\bar{\imath}\colon LC_c \to LC$ and $Q\colon LC \to LC_c$, respectively. Then the composite $\mathbf{L}(Fi)Q\colon LC \to LD$ is a total left derived functor of F.*

Remark. Since i and Fi preserve all weak equivalences, their total (left) derived functors $\bar{\imath}$ and $\mathbf{L}(Fi)$ exist by the preceding lemma. By Lemma 1.24 citing [Cis19, Theorem 7.5.18], $\bar{\imath}$ is an equivalence of ∞-categories, so that there exists a quasi-inverse Q.

Proof of proposition. As a quasi-inverse of an equivalence of ∞-categories, $\bar{\imath}$ is left adjoint to Q,[1] so that $\bar{\imath}^*$ is naturally isomorphic to $Q_!$ as functors from $\mathrm{Fun}(LC, LD)$ to $\mathrm{Fun}(LC_c, LD)$.[2] Therefore, $\bar{\imath}^*$ is left adjoint to Q^*, so that $\mathbf{L}(Fi)Q = Q^*(\mathbf{L}(Fi))$ represents the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(LC_c, LD)}\left(\bar{\imath}^*(-), \mathbf{L}(Fi)\right) : \mathrm{Fun}(LC, LD)^{\mathrm{op}} \longrightarrow \mathcal{S},$$

which is the composite of the functor represented by $\mathbf{L}(Fi)$ and the dual of $\bar{\imath}^*\colon \mathrm{Fun}(LC, LD) \to \mathrm{Fun}(LC_c, LD)$. Per definition, the former is (naturally isomorphic to) the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C_c, LD)}\left((\gamma_{C_c})^*(-), \gamma_D Fi\right) : \mathrm{Fun}\left(LC_c, LD\right)^{\mathrm{op}} \longrightarrow \mathcal{S},$$

which itself is naturally isomorphic to the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C, LD)}\left((\gamma_C)^*\left(Q^*(-)\right), \gamma_D F\right) : \mathrm{Fun}\left(LC_c, LD\right)^{\mathrm{op}} \longrightarrow \mathcal{S}.$$

For a proof of the last natural isomorphism, we would like to refer the reader to [Cis19, Corollary 7.5.17], which proves the dual (and global) version of this statement.[3] So, $\mathbf{L}(Fi)Q$ represents the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C, LD)}\left((\gamma_C)^*\left(Q^*\left(\bar{\imath}^*(-)\right)\right), \gamma_D F\right) : \mathrm{Fun}\left(LC, LD\right)^{\mathrm{op}} \longrightarrow \mathcal{S},$$

which is naturally isomorphic to

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C, LD)}\left((\gamma_C)^*(-), \gamma_D F\right) : \mathrm{Fun}\left(LC, LD\right)^{\mathrm{op}} \longrightarrow \mathcal{S}$$

because $Q^*\bar{\imath}^*$ is naturally isomorphic to the identity. Hence, $\mathbf{L}(Fi)Q$ is a total left derived functor of F. $\square$

Corollary 2.11. *[Cis06, Proposition 7.5.28] Let $F\colon C \to D$ be a right exact functor between quasi-categories with weak equivalences and cofibrations. Then the total left derived functor $\mathbf{L}F\colon LC \to LD$ is right exact.*

Proof. Since equivalence of ∞-categories preserve finite limits, the preceding proposition and Lemma 2.4 imply that $\mathbf{L}F$ is the composite of two right exact functors. $\square$

Definition 2.12. [Szu14, Definition 1.9] A *fibration* in CofQCat is a right exact functor $F\colon C \to D$ between quasi-categories with weak equivalences and cofibrations that satisfies the following properties:

0. F is an *inner fibration*: For any pair of inner horn $\tau\colon \Lambda^n_k \to C$ and simplex $\sigma\colon \Delta^n \to D$ such that $F(\tau) = \sigma|_{\Lambda^n_k}$, there is a simplex $\ell\colon \Delta^n \to C$ such that $\ell|_{\Lambda^n_k} = \tau$ and $F(\ell) = \sigma$.

1. F is an *isofibration*: For any isomorphism $g\colon F(c_0) \to d$ in D with c_0 an object in C, there is an isomorphism $f\colon c_0 \to c_1$ in C with $F(f) = g$.

[1] See [Cis19, Proposition 6.1.6]

[2] See [Cis19, Theorem 6.1.23].

[3] In order to apply this reference, D needs to be $\mathbf{U}$-complete, which is generally not the case, but, as D can be replaced by $\mathrm{Fun}\left(D^{\mathrm{op}}, \mathcal{S}\right)$ via the continuous, fully faithful Yoneda embedding, we may assume that D is $\mathbf{U}$-complete.

2. *F lifts factorizations*: For any map $f\colon c_0 \to c_1$ in C with c_0 cofibrant and any commutative triangle β in D of the form

$$\begin{tikzcd} & d \arrow[dr, "t"] & \\ F(c_0) \arrow[ur, "j"] \arrow[rr, "F(f)"'] & & F(c_1) \end{tikzcd}$$

with j a cofibration and t a weak equivalence in D, there is a commutative triangle α in C of the form

$$\begin{tikzcd} & c \arrow[dr, "s"] & \\ c_0 \arrow[ur, "i"] \arrow[rr, "f"'] & & c_1 \end{tikzcd}$$

with i a cofibration and s a weak equivalence in C such that $F(\alpha) = \beta$.

3. *F lifts pseudo-factorizations*: For any map $f\colon c_0 \to c_1$ in C with c_0 cofibrant and any commutative square β in D of the form

$$\begin{tikzcd} F(c_0) \arrow[r, "F(f)"] \arrow[d, "j"'] & F(c_1) \arrow[d, "w"] \\ d_0 \arrow[r, "t"'] & d_1 \end{tikzcd}$$

with j a cofibration, t a weak equivalence and w a trivial cofibration in D, there is a commutative square α in C of the form

$$\begin{tikzcd} c_0 \arrow[r, "f"] \arrow[d, "i"'] & c_1 \arrow[d, "v"] \\ c_2 \arrow[r, "s"'] & c_3 \end{tikzcd}$$

with i a cofibration, s a weak equivalence and v a trivial cofibration in C such that $F(\alpha) = \beta$.

4. *F reflects cofibrant objects*: Any object c in C for which $F(c)$ is cofibrant in D is already cofibrant.

5. *F reflects the 2-out-of-6 property for weak equivalences between cofibrant objects*: For any a triple (f, g, h) of composable maps between cofibrant objects in C such that there are composites $g \circ f$ and $h \circ g$ which are weak equivalences in C, a composite $h \circ g \circ f$ is a weak equivalence in C if $F(h \circ g \circ f)$ is a weak equivalence in D.

   By the 2-out-of-3 property for weak equivalences, this would imply that all maps f, g and h are weak equivalences.

Remark. One can easily check that all identities of cofibration quasi-categories are fibrations and that the class of fibrations of quasi-categories with weak equivalences and cofibrations is closed under composition.

Definition 2.13. Let $\mathrm{CofQCat}_f$ be the full subcategory of CofQCat spanned by the fibrant objects.

The Main Theorem of Section 2

For unfortunate reasons, it is not possible (for the author) to show the factorization axiom for CofQCat, but for $\mathrm{CofQCat}_f$. Although the remaining conditions and axioms of Lemma 1.1 are going to be proven for CofQCat, we would like to state the following weaker assertion:

Theorem 2.14. *The nerve of* $\mathrm{CofQCat}_f$ *is a homotopical quasi-category of fibrant objects. For reference, the single conditions are listed as follows:*

- *Lemma 2.13 for the fibrancy of all objects;*
- *Remark of Lemma 2.12 for the stability of fibrations under composition and for the fact that all identities are fibrations;*
- *Lemma 2.6 for the stability of weak equivalences under identities;*
- *Lemma 2.15 for the 2-out-of-3 property for weak equivalences;*
- *Lemma 2.16 for the existence of a terminal object;*
- *Lemma 2.24 for the pullback axioms for cofibrations and trivial cofibrations;*
- *Lemma 2.26 for the factorization axiom.*

Remark. By (the dual of) Lemma 1.27, we can use the equivalence of categories between CofQCat and FibQCat, which is given by forming the opposite or dual quasi-categories, in order to transfer the notion of weak equivalence and fibrations from CofQCat onto FibQCat such that the nerve of a specific subcategory of FibQCat is a homotopical quasi-category of fibrant objects, namely the subcategory of fibrant objects.

Proposition 2.15. *The class of weak equivalences in* CofQCat *satisfies the 2-out-of-6 property.*

Proof. Let $F\colon C \to D$ and $G\colon D \to E$ be two right exact functors between quasi-categories with weak equivalences and cofibrations. By [Cis19, Proposition 7.5.29], the following diagram of ∞-categories

$$\begin{tikzcd} & LC \arrow[dr, "\mathbf{L}G"] & \\ LD \arrow[ur, "\mathbf{L}F"] \arrow[rr, "\mathbf{L}(GF)"'] & & LE \end{tikzcd}$$

commutes up to natural isomorphism. By [Cis19, Proposition 6.1.6 & 6.1.7] functors that are naturally isomorphic to equivalences of ∞-categories are themselves equivalences, so that we may conclude this proof by the 2-out-of-6 property for equivalences of ∞-categories. $\square$

Proposition 2.16. CofQCat *admits a terminal object, namely* Δ^0.

Proof. Δ^0 carries a unique structure of a quasi-category with weak equivalences and cofibrations: As identities have to be cofibrations, so must be the identity of 0. Therefore, 0 is cofibrant and, thus, the identity of 0 is a trivial cofibration by Lemma 1.6. Then the statement follows from the fact that, for any quasi-category with weak equivalences and cofibrations C, the canonical functor $C \to \Delta^0$ is already right exact. □

Remark. The fibrant objects in CofQCat are those quasi-categories with weak equivalences and cofibrations that satisfy the following conditions:

0. The underlying simplicial set is a quasi-category.
1. Each object admits an object that is isomorphic to it.
2. Each map with cofibrant source factors into a cofibration followed by a weak equivalence.
3. Each map with cofibrant source pseudo-factors into a cofibration followed by a zig-zag of a weak equivalence and a trivial cofibration.
4. All objects are cofibrant.
5. Weak equivalences (between cofibrant objects) satisfy the 2-out-of-6 property.

Note that any quasi-category with weak equivalences and cofibrations satisfies all, but the last two conditions: (0) obviously holds. For (1), one may choose the identity map, and (2) is the factorization axiom and already implies condition (3).

For this reason, the fibrant objects in $\mathrm{CofQCat}_f$ will be called *homotopical quasi-categories of cofibrant objects*, and we would like to mention that this is the quasi-categorical analogue to the *cofibration category* in [Szu14].

Example 2.17. A notable example of homotopical quasi-categories of cofibrant objects are localizations of quasi-categories with weak equivalences and cofibrations. In particular, any object in CofQCat has a fibrant replacement.

The Pullback Axioms

In order to prove the pullback axioms for CofQCat, it will be helpful to detect fibrations in CofQCat by right lifting properties in an ambient category.

Definition 2.18. Let the category $\overline{\mathrm{CofQCat}_f}$ be defined by the following data:

- The objects are quasi-categories with two classes of maps, the so-called *cofibrations* and *weak equivalences*, which contain all identities and are closed under composition.
- The morphisms are functors between quasi-categories that preserve both classes of maps.
- The composition law is given by the usual composition of functors between quasi-categories.

Remark. Because right exact functors between homotopical quasi-categories of cofibrant objects preserve all weak equivalences and cofibrations, $\mathrm{CofQCat}_f$ is a (non-full) subcategory of $\overline{\mathrm{CofQCat}}$.

Example 2.19. Let k and n be natural numbers such that $0 < k < n$. Then only the identity maps in the inner horn Λ^n_k and the simplex Δ^n are declared to be weak equivalences and cofibrations.

Let $[n]$ denote the canonically totally ordered set $\{0, \ldots, n\}$. Moreover, we regard partially ordered sets as skeletal thin categories[1], so that monotone (increasing) functions between them become functors between their categories. Then the examples of our interest are:

- $[0]$ where the identity of 0 is a trivial cofibration;
- $[1]$ where the identities are trivial cofibrations;
- $[2]$ where $0 \to 1$ is a cofibration and $1 \to 2$ is a weak equivalence.

For Lemma 2.21, we would like to modify the structure on $[1]$:

- $[1]_w$ has additionally $0 \to 1$ as a weak equivalence;
- $[1]_c$ has additionally $0 \to 1$ as a cofibration.

Let $\Box$ denote the power set of $\{0, 1\}$, partially ordered by the $\subseteq$-relation. We declare $\emptyset \subseteq \{0\}$ to be a cofibration, $\{0\} \subseteq \{0, 1\}$ to be a weak equivalence and $\{1\} \subseteq \{0, 1\}$ to be both.

Moreover,

Lastly, let J denote the *free-standing isomorphism*, i.e. the category with two objects 0 and 1 and with two non-trivial maps $0 \to 1$ and $1 \to 0$, which are mutually inverse to each other. Then all maps in J are declare to be trivial cofibrations.

Lemma 2.20. *Let $F \colon C \to D$ be a right exact functor between homotopical quasi-categories of cofibrant objects. Then F is a fibration if and only if F has, in $\overline{\mathrm{CofQCat}}_f$, the right lifting property with respect to (the nerve of) the following functors:*

0. $\Lambda^n_k \hookrightarrow \Delta^n$;
1. $[0] \to J$ *sending 0 to 0;*
2. $[1] \to [2]$ *sending 0 to 0 and 1 to 2;*
3. $[1] \to \Box$ *sending 0 to $\emptyset$ and 1 to $\{1\}$.*

Here, the classes of weak equivalences and cofibrations are defined as in the preceding example.

Remark. The proof is left to the reader, but it should be noted that condition (4) and (5) of Lemma 2.12 already hold for F as all objects in C are cofibrant and the weak equivalences in C satisfy the 2-out-of-6 property.

[1] In our examples, categories themselves are regarded as quasi-categories via the nerve functor.

Lemma 2.21. *[Szu14, Proposition 1.11] Let $P\colon C \to D$ be a fibration of homotopical quasi-categories of cofibrant objects. Then P is a weak equivalences if and only if P has, in $\overline{\mathrm{CofQCat}_f}$, the right lifting property with respect to the nerve of the following functors:*

1. *$[1] \to [1]_w$ sending 0 to 0 and 1 to 1;*
2. *$[0] \to [1]_c$ sending 0 to 0.*

We will use the following characterization of weak equivalences of quasi-categories with weak equivalences and cofibrations:

Theorem 2.22. *[Cis19, Proposition 7.6.15] A right exact functor $F\colon C \to D$ between quasi-categories with weak equivalences and cofibrations is a weak equivalence if and only if it satisfies the following properties:*

1. *For any map $f\colon a \to b$ between cofibrant objects in C for which $F(f)$ is a weak equivalence in D, there is a map $g\colon b \to c$ in C with c cofibrant such that a composite $g \circ f\colon a \to c$ is a weak equivalence in C.*
2. *For any cofibrant objects c_0 in C and d_0 in D and any map $g\colon F(c_0) \to d_0$ in D, there is a commutative diagram in D of the form*

$$\begin{tikzcd} F(c_0) \arrow[r, "F(f)"] \arrow[d, "g"'] & F(c_1) \arrow[d, "s"] \\ d_0 \arrow[r, "t"'] & d_1 \end{tikzcd}$$

where $f\colon c_0 \to c_1$ is a map in C with c_1 cofibrant and both $s\colon F(c_1) \to d_1$ and $t\colon d_0 \to d_1$ are weak equivalences in D with d_1 cofibrant.

Proof of lemma. First, assume that P has the right lifting properties with respect to $[1] \to [1]_w$ and $[0] \to [1]_c$ in $\overline{\mathrm{CofQCat}_f}$. In particular, P reflects weak equivalences and it remains to show that P satisfies the second condition of Lemma 2.22. So, consider a map $g\colon P(c_0) \to d_0$ in D with c_0 and d_0 cofibrant. As right exact functors preserves cofibrant objects, $P(c_0)$ is cofibrant in D, so that, by the factorization axiom, there is a commutative triangle in D

$$\begin{tikzcd} & d_1 \arrow[dr, "s"] & \\ P(c_0) \arrow[ur, "i"] \arrow[rr, "g"'] & & d_0 \end{tikzcd}$$

with i a cofibration and s a weak equivalence. By the right lifting property with respect to $[0] \to [1]_c$, there is a cofibration $f\colon c_0 \to c_1$ in C with $P(f) = i$. Finally, we may conclude with $t := 1_{d_0}$ as then the diagram in D

$$\begin{tikzcd} P(c_0) \arrow[r, "F(f)"] \arrow[d, "g"'] & P(c_1) \arrow[d, "s"] \\ d_0 \arrow[r, "t"'] & d_1 \end{tikzcd}$$

commutes per construction.

Conversely, assume that P is a weak equivalence in CofQCat, i.e. P satisfies condition (1) and (2) of Lemma 2.22. Let $f\colon a \to b$ be a map between cofibrant objects in C such that $P(f)$ is a weak equivalence in D. By condition (1), there is a map $g\colon b \to c$ in C with c cofibrant such that a composition $g \circ f\colon a \to c$ is a weak equivalence in C. As right exact functors preserve weak equivalences between cofibrant objects, $P(g \circ f)$ is a weak equivalence in D and, by the 2-out-of-3 property for weak equivalences, $P(g)$ is a weak equivalence in D. By condition (1), there is a map $h\colon c \to d$ in C such that $h \circ g\colon b \to d$ is a weak equivalence in C, so that $P(h \circ g)$ and thus $P(h)$ are weak equivalences in D. Hence, $P(h \circ g \circ f)$ is a weak equivalence in D, as a composition of weak equivalences. As P reflects the 2-out-of-6 property for weak equivalences between cofibrant objects, $h \circ g \circ f$ and, by the 2-out-of-3 property, f, g and h are weak equivalences. In particular, we have shown that f is a weak eqivalence in C if and only if $P(f)$ is a weak equivalence in D. Therefore, P has the right lifting property with respect to $[1] \to [1]_w$.

Now, let c_0 be a cofibrant object in C and $i\colon P(c_0) \to d_0$ a cofibration in D with d_0 cofibrant. By Lemma 2.22, there is a commutative diagram in D

$$\begin{tikzcd} P(c_0) \arrow[r, "P(f)"] \arrow[d, "i"'] & P(c_1) \arrow[d, "s"] \\ d_0 \arrow[r, "t"'] & d_1. \end{tikzcd}$$

where $f\colon c_0 \to c_1$ is a map in C with c_1 cofibrant and both $s\colon P(c_1) \to d_1$ and $t\colon d_0 \to d_1$ are weak equivalences in D with d_1 cofibrant. By the pushout axiom for cofibrations, there is a pushout square of D

$$\begin{tikzcd} P(c_0) \arrow[r, "P(f)"] \arrow[d, "i"'] & P(c_1) \arrow[d, "\iota_2"] \\ d_0 \arrow[r, "\iota_1"'] & d_0 \sqcup_{P(c_0)} P(c_1) \end{tikzcd}$$

with ι_2 a cofibration, so that the previous commutative square induces a map $(t,s)\colon d_0 \sqcup_{P(c_0)} P(c_1) \to d_1$. As the canonical map $\varnothing_C \to d_0 \sqcup_{P(c_0)} P(c_1)$ can be factored into $\varnothing_C \to P(c_1)$ followed by ι_2, both of which cofibrations, $d_0 \sqcup_{P(c_0)} P(c_1)$ is a cofibrant object in D and, by the factorization axiom, there is a commutative triangle in D

$$\begin{tikzcd} & d \arrow[dr, "r"] & \\ d_0 \sqcup_{P(c_0)} P(c_1) \arrow[ur, "j"] \arrow[rr, "{(t,s)}"'] & & d_1 \end{tikzcd}$$

where j is a cofibration and r is a weak equivalence. This induces a commutative triangle in D

$$\begin{tikzcd} & d \arrow[dr, "r"] & \\ d_0 \arrow[ur, "j\circ\iota_1"] \arrow[rr, "{(t,s)\circ\iota_1}"'] & & d_1 \end{tikzcd}$$

Because of $t = (t, s)\circ\iota_1$, which is a a weak equivalence, $j\circ\iota_1$ is a weak equivalence by the 2-out-of-3 property. As a composition of two cofibrations, $j \circ \iota_2$ is a cofibration and fits into the commutative triangle

$$\begin{tikzcd} & d \arrow[dr, "r"] & \\ P(c_1) \arrow[ur, "j\circ\iota_2"] \arrow[rr, "(t,s)\circ\iota_2"'] & & d_1. \end{tikzcd}$$

As we have $s = (t, s)\circ\iota_2$, which is a weak equivalence, $j\circ\iota_2$ is a weak equivalence by the 2-out-of-3 property, thus a trivial cofibration. Hence, the square in D

$$\begin{tikzcd} P(c_0) \arrow[r, "P(f)"] \arrow[d, "i"'] & P(c_1) \arrow[d, "j\circ\iota_2"] \\ d_0 \arrow[r, "j\circ\iota_1"'] & d, \end{tikzcd}$$

which commutes per construction, is a pseudo-factorization of $P(f)$. Since P lifts pseudo-factorizations, there is a cofibration i' in C with $P(i') = i$. This shows that P has the right lifting property with respect to $[0] \to [1]_c$. □

Remark. For future references, we would also like to highlight the following statement: Let $F\colon C \to D$ be a right exact functor between quasi-categories with weak equivalences and cofibrations and assume that the class fo weak equivalences between cofibrant objects in C satisfies the 2-out-of-6 property. Then F satisfies the first condition of Lemma 2.22 citing [Cis19, Proposition 7.6.15] if and only if F reflects weak equivalences between cofibrant objects. The "if"-direction trivially holds, and the "only if"-direction has already been implicitly shown in the second to last paragraph of the above proof.

Similarly to the last paragraph of the proof, we can show that, given a commutative diagram in D of the form

$$\begin{tikzcd} F(c_0) \arrow[r, "F(f)"] \arrow[d, "g"'] & F(c_1) \arrow[d, "s"] \\ d_0 \arrow[r, "t"'] & d_1 \end{tikzcd}$$

as in the second condition of Lemma 2.22, we may assume without loss of generality that f and t are additionally cofibrations: First, factorize f into a cofibration $f'\colon c_0 \to c_1'$ followed by a weak equivalence $r\colon c_1' \to c_1$, and form the pushout square of the form

$$\begin{tikzcd} F(c_0) \arrow[r, "F(f')"] \arrow[d, "g"'] & F(c_1') \arrow[d, "g'"] \\ d_0 \arrow[r, "i"'] & d_1' \end{tikzcd}$$

with i a cofibration. Then there is an induced map $(t, sF(r))\colon d_1' \to d_1$ such that the triangles

$$\begin{tikzcd} & d_1' \arrow[dr, "{(t,sF(r))}"] & \\ d_0 \arrow[ur, "i"] \arrow[rr, "t"'] & & d_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & d_1' \arrow[dr, "{(t,sF(r))}"] & \\ F(c_1') \arrow[ur, "g'"] \arrow[rr, "s"'] & & d_1 \end{tikzcd}$$

commute. Now, factorize $(t, sF(r))$ into a cofibration $j\colon d_1' \to d_1''$ followed by a weak equivalence $u\colon d_1'' \to d_1$. Note that $sF(r)$ is a weak equivalence, as a composite of such, and that the triangles

$$\begin{tikzcd} & d_1'' \arrow[dr, "u"] & \\ d_0 \arrow[ur, "ji"] \arrow[rr, "t"'] & & d_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & d_1'' \arrow[dr, "u"] & \\ F(c_1') \arrow[ur, "jg'"] \arrow[rr, "sF(r)"'] & & d_1 \end{tikzcd}$$

commute, so that ji and jg' are weak equivalences by the 2-out-of-3 property. As a composite of cofibrations, ji is one itself. Hence,

$$\begin{tikzcd} F(c_0) \arrow[r, "F(f')"] \arrow[d, "g"'] & F(c_1') \arrow[d, "jg'"] \\ d_0 \arrow[r, "ji"'] & d_1'' \end{tikzcd}$$

is a commutative square with f' a cofibration, ji a trivial cofibration and jg' a weak equivalence.

Proposition 2.23. *Let $F\colon A \to B$ and $G\colon B \to C$ be morphisms in* CofQCat *such that all objects in A, B and C are cofibrant, G lifts factorizations of maps (with cofibrant source) and F or G are inner fibrations and isofibrations. Then the span*

$$A \xrightarrow{F} B \xleftarrow{G} C$$

admits a pullback in CofQCat*. In fact, we will show that any pullback in* sSet *of the underlying simplicial sets can be equipped with the structure of a homotopical quasi-category of cofibrant objects such that it is a pullback in* CofQCat*.*

Remark. Instead of the condition that all objects in A, B and C are cofibrant, we may impose the following:

- F and G preserve all cofibrations and weak equivalences.
- coF or coG is an inner fibration and an isofibration.
- wF or wG is an inner fibration and an isofibration.

In particular, this implies that CofQCat has all finite products.

Proof of proposition. Since F or G are inner fibrations and isofibrations between quasi-categories, that is, fibrations in the Joyal model structure on sSet, there is a pullback square of the form

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "G"] \\ A \arrow[r, "F"'] & B \end{tikzcd} \tag{2.23.1}$$

in sSet. Since all objects in A, B and C are cofibrant, F and G preserve all cofibrations per definition and all weak equivalences by Ken Brown's lemma, so that there are spans of quasi-categories of the form

$$coA \xrightarrow{F} coB \xleftarrow{G} coC \quad \text{and} \quad wA \xrightarrow{F} wB \xleftarrow{G} wC \ .$$

Since all objects are cofibrant and all isomorphisms between cofibrant objects are trivial cofibrations by Lemma 1.6, one can check that, in each of the two spans, at least one leg is an inner fibration and an isofibration, so that there are pullback squares of the form

$$\begin{tikzcd} co\,(A \times_B C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & coC \arrow[d, "G"] \\ coA \arrow[r, "F"'] & coB \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} w\,(A \times_B C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & wC \arrow[d, "G"] \\ wA \arrow[r, "F"'] & wB \end{tikzcd}$$

in sSet. Easily, one can check that the canonical maps $co\,(A \times_B C) \to A \times_B C$ and $w\,(A \times_B C) \to A \times_B C$ are inclusions of subcategories of $A \times_B C$.

Now, we show that $(A \times_B C, w\,(A \times_B C)\,, co\,(A \times_B C))$ forms a quasi-category with weak equivalences and cofibrations. First, identities in $A \times_B C$ are given by a pair of identities in A and C which F and, respectively, G send to the same identity in B. Since identities in A, B and C are cofibrations, so are identities in $A \times_B C$. Second, commutative triangles in $A \times_B C$ are analogously given by commutative triangles in A and C which F and G send to the same commutative triangle in B, so that the 2-out-of-3 property for $w\,(A \times_B C)$ follows from that for wA, wB and wC and the fact that F and G preserve all weak equivalences since all objects are cofibrant. Third, since dualization forms an involution of sSet, thus an equivalence of categories of the form sSet $\to$ sSet, the dual diagram of (2.23.1)

$$\begin{tikzcd} (A \times_B C)^{\mathrm{op}} \arrow[r, "(\pi_2)^{\mathrm{op}}"] \arrow[d, "(\pi_1)^{\mathrm{op}}"'] & C^{\mathrm{op}} \arrow[d, "G^{\mathrm{op}}"] \\ A^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"'] & B^{\mathrm{op}} \end{tikzcd}$$

is still a pullback square in sSet. In fact, F^{op} or G^{op} are still isofibrations, so that this pullback square presents a pullback square of ∞-categories. Therefore, the existence of an initial object and the pushout axioms for cofibrations and trivial cofibrations in $A \times_B C$ follow from an application of [Cis19, Proposition 6.2.19] as initial objects and pushouts of (trivial) cofibrations are limits of diagrams in $(A \times_B C)^{\mathrm{op}}$, which F^{op} and G^{op} are assumed to preserve by right exactness. In particular, this reference states that $(\pi_1)^{\mathrm{op}}$ and $(\pi_2)^{\mathrm{op}}$ preserves said limits, so that π_1 and π_2 are right exact since cofibrations and weak equivalences are preserved by construction. It should also be mentioned that, because the projections preserve initial objects, one can quickly check that all objects in $A \times_B C$ are cofibrant. Lastly, we want to verify the factorization axiom for $A \times_B C$. Let $f \colon x \to y$ be a map in $A \times_B C$ (with cofibrant source). By the factorization axiom in A, there is a commutative triangle in A of the form

$$\begin{tikzcd} & a \arrow[dr, "s"] & \\ \pi_1(x) \arrow[ur, "i"] \arrow[rr, "\pi_1(f)"'] \arrow[rr, phantom, shift left=3ex, "\sigma"] & & \pi_1(y) \end{tikzcd}$$

with i a cofibration and s a weak equivalence in A. Since F preserves all cofibrations and weak equivalences, the induced commutative square in B

$$\begin{tikzcd} & F(a) \arrow[dr, "F(s)"] & \\ F(\pi_1(x)) \arrow[ur, "F(i)"] \arrow[rr, "F(\pi_1(f))"'] \arrow[rr, phantom, shift left=3ex, "F(\sigma)"] & & F(\pi_1(y)) \end{tikzcd}$$

is a factorization of $F(\pi_1(f))$ into a cofibration $F(i)$ followed by a weak equivalence $F(s)$. Because of $F(\pi_1(f)) = G(\pi_2(f))$ and because G lifts factorizations, there is a commutative triangle in C of the form

$$\begin{tikzcd} & c \arrow[dr, "t"] & \\ \pi_2(x) \arrow[ur, "j"] \arrow[rr, "\pi_2(f)"'] \arrow[rr, phantom, shift left=3ex, "\tau"] & & \pi_2(y) \end{tikzcd}$$

with j a cofibration and t a weak equivalence in A such that $G(\tau) = F(\sigma)$. Thus, the commutative triangles σ in A and τ in C define a factorization of f into a cofibration $(i, j)\colon x \to (c, d)$ followed by a weak equivalence $(s, t)\colon (c, d) \to y$ in $A \times_B C$.

Finally, we can show that $A \times_B C$ is a pullback in CofQCat. Let

$$\begin{tikzcd} D \arrow[r, "H_2"] \arrow[d, "H_1"'] & C \arrow[d, "G"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

be a commutative square in CofQCat. By the universal property of the underlying quasi-categories, there is a unique functor $H\colon D \to A \times_B C$ such that $H\pi_i = H_i$ holds for $i \in \{1, 2\}$. Since H_1 and H_2 are right exact, there are induced commutative squares of quasi-categories of the form

$$\begin{tikzcd} wD_c \arrow[r, "H_2"] \arrow[d, "H_1"'] & wC \arrow[d, "G"] \\ wA \arrow[r, "F"'] & wB \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} coD_c \arrow[r, "H_2"] \arrow[d, "H_1"'] & coC \arrow[d, "G"] \\ coA \arrow[r, "F"'] & coB \end{tikzcd}$$

which correspond to functors $wD_c \to w\,(A \times_B C)$ and $coD_c \to co\,(A \times_B C)$ by the universal property of pullbacks in sSet. One can check that these induced functors are restrictions of H, i.e. H preserves cofibrations and weak equivalences of cofibrant objects. The fact that H preserves intial objects and pullbacks of cofibrations follow from an application of [Cis19, Proposition 6.2.19]:

Let $\delta\colon I \to D^{\mathrm{op}}$ be either the empty diagram or a cospan of cofibrant objects where one leg is a cofibration in D. By the existence of initial objects and the pushout axiom for cofibrations, the limits of δ and $(H_i)^{\mathrm{op}}(\delta)$ for $i \in \{1,2\}$ exist. By right exactness, these limits are preserved by F^{op} and G^{op}, so that [Cis19, Proposition 6.2.19] implies that $H^{\mathrm{op}}(\delta)$ has a limit in $(A \times_B C)^{\mathrm{op}}$ whose image by $(\pi_i)^{\mathrm{op}}$ is a limit of $(H_i)^{\mathrm{op}}(\delta)$ for $i \in \{1,2\}$, which itself is the image by $(H_i)^{\mathrm{op}}$ of the limit of δ by right exactness of H_i. Thus, the limit of $H^{\mathrm{op}}(\delta)$ is the image by H^{op} of the limit of δ, i.e. H preserves initial objects and pullbacks of cofibrations. Therefore, H is right exact and this concludes the proof that $A \times_B C$ is a pullback in CofQCat. □

Corollary 2.24 (Pullback axioms in CofQCat)**.** *Let $F\colon A \to B$ and $G\colon B \to C$ be right exact functors between homotopical quasi-categories of cofibrant objects such that G is a (trivial) fibration. Then the span in* CofQCat

$$A \xrightarrow{F} B \xleftarrow{G} C$$

admits a pullback in CofQCat*, and, for any pullback square in* CofQCat *of the form*

$$\begin{tikzcd} D \arrow[r, "F'"] \arrow[d, "G'"'] & C \arrow[d, "G"] \\ A \arrow[r, "F"'] & B, \end{tikzcd}$$

G' is a (trivial) fibration.

Proof. The preceding proposition implies the first assertion. The second to last paragraph of the preceding proof also shows that the specific pullback $A \times_B C$ in CofQCat is a pullback in $\overline{\mathrm{CofQCat}_f}$ as well. As classes of morphisms defined by right lifting properties are closed under pullbacks[1], $\pi_1\colon A \times_B C \to A$ is a fibration (or trivial fibration) by Lemma 2.20 (or Lemma 2.21, respectively). By the universal property, there is a canonical isomorphism $I\colon D \to A \times_B C$ in CofQCat such that $\pi_1 I = G'$ and $\pi_2 I = F'$. Given any object d in D, let $\varnothing_D \to d$ denote the canonical map in D. Then its image under I is $\varnothing_{A\times_B C} \to I(d)$, which is a cofibration since any object in $A \times_B C$ is cofibrant, so that its image under the inverse I^{-1} is a cofibration as well, but this has to be $\varnothing_D \to d$ again. Thus, d was cofibrant. Therefore, all objects in D are cofibrant. Recall that right exact functors whose domain consists of only cofibrant objects preserve all cofibrations and weak equivalences, so that the diagram

$$\begin{tikzcd} D \arrow[r, "I"] \arrow[d, "G'"] & A \times_B C \arrow[r, "I^{-1}"] \arrow[d, "\pi_1"] & D \arrow[d, "G'"] \\ A \arrow[r, "\mathrm{id}_A"] & A \arrow[r, "\mathrm{id}_A"] & A \end{tikzcd}$$

is a diagram in $\overline{\mathrm{CofQCat}_f}$ and is clearly commutative. Hence, G' is a retract of π_1 and has the same right lifting properties as π_1 because classes of morphisms defined by right lifting properties are closed under retracts. By Lemma 2.20 or Lemma 2.21, G' is a fibration or a trivial fibration, respectively. □

[1] See [Cis19, Proposition 2.1.4 (g)].

Using Lemma 2.22 citing [Cis19, Proposition 7.6.15], we can already prove Lemma 1.16 for CofQCat without the use of the factorization axiom for $\mathrm{CofQCat}_f$.

Proposition 2.25. *Consider a pullback square in* CofQCat *of the form*

$$\begin{tikzcd} D \arrow[r, "F'"] \arrow[d, "G'"'] & C \arrow[d, "G"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

with G a fibration, F a weak equivalence and A and B fibrant in CofQCat. *Then F' is a weak equivalence. In particular, such a pullback square is a homotopy pullback square.*

Proof. We will verify the two conditions of Lemma 2.22, modified according to its remark:

First condition Let $f\colon a \to b$ be a map of cofibrant objects in D such that $F'(f)$ is a weak equivalence in C. Then $G(F'(f))$ is a weak equivalence in B. Because F is a weak equivalence in CofQCat and because of $GF' = FG'$, this implies that $G'(f)$ is a weak equivalence in A. Therefore, f is a weak equivalence in D.

Second condition Let $g\colon F'(a) \to x$ be a map in C with a an object in D. Then there is a cofibration $h\colon G'(a) \to c$ in A and a commutative square in B fo the form

$$\begin{tikzcd} G(F'(a)) \arrow[r, "F(h)"] \arrow[d, "G(g)"'] & F(c) \arrow[d, "v"] \\ G(x) \arrow[r, "u"'] & z \end{tikzcd}$$

with u a trivial cofibration and v weak equivalences in B. Note that we used $GF' = FG'$ here, and that this diagram forms a pseudo-factorization of $G(g)$. Since G is a fibration in CofQCat, it lifts this pseudo-factorization to one of g in C, i.e. there is a commutative square in C of the form

$$\begin{tikzcd} F'(a) \arrow[r, "k"] \arrow[d, "g"'] & d \arrow[d, "s"] \\ x \arrow[r, "t"'] & y \end{tikzcd}$$

with k a cofibration, t a trivial cofibration and s a weak equivalence that G sends to the diagram above. Note that h and k define a cofibration $f\colon a \to b$ in D. Hence, k is of the form $F'(f)\colon F'(a) \to F'(b)$. This concludes the proof. $\square$

The Factorization Axiom and Path Objects

We will show that, given a right exact functor $F\colon C \to D$ between homotopical quasi-categories of cofibrant objects, any factorization of its total (left) derived functor provides a factorization of F via pullback. For this, we will refer to constructions and statements in Section 3.

Proposition 2.26 (Factorization axiom for $\mathrm{CofQCat}_f$)**.** *Let $F\colon C \to D$ be a right exact functor between homotopical quasi-categories of cofibrant objects. Then there exists a factorization of F into a weak equivalence followed by a fibration, i.e. there is a commutative square in* CofQCat *of the form*

$$\begin{tikzcd} & C' \arrow[dr, "P"] & \\ C \arrow[ur, "S"] \arrow[rr, "F"'] & & D \end{tikzcd}$$

with $S\colon C \to C'$ and $P\colon C' \to D$ a fibration.

Proof. Since all objects in C and D are cofibrant, F preserves all weak equivalences, so that there is a commutative square in sSet of the form

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, "\bar{F}"'] & LD \end{tikzcd}$$

with γ_C and γ_D localization functors. A construction of such a commutative square will be given in Lemma 3.10. Note that all functors are right exact, so that this diagram is a commutative square in CofQCat as well. If seen as fibrant objects in the Joyal model structure on sSet, there is a factorization of $\bar{F}$ into a weak categorical equivalence followed by a functor which is both an inner fibration and an isofibration. We claim that this forms a factorization in CofQCat as well. In Lemma 3.7, we will prove the slightly more general statement that any factorization of a right exact functor between finitely cocomplete quasi-categories in the Joyal model structure forms a factorization in CofQCat. Thus, there is a commutative triangle in CofQCat of the form

$$\begin{tikzcd} & X \arrow[dr, "Q"] & \\ LC \arrow[ur, "T"] \arrow[rr, "\bar{F}"'] & & LD \end{tikzcd}$$

with S a weak equivalence and P a fibration in CofQCat. By Lemma 2.24, there is a pullback square in CofQCat of the form

$$\begin{tikzcd} C' \arrow[r, "P"] \arrow[d, "G"'] & D \arrow[d, "\gamma_D"] \\ X \arrow[r, "Q"'] & LD \end{tikzcd}$$

with P a cofibration and, by Lemma 2.25, G a weak equivalence. Since the diagram in CofQCat

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "T\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ X \arrow[r, "Q"'] & LD \end{tikzcd}$$

commutes, there is a right exact functor $S\colon C \to C'$ such that the triangles in CofQCat

$$\begin{tikzcd} & C' \arrow[dr, "G"] & \\ C \arrow[ur, "S"] \arrow[rr, "T\gamma_C"'] & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C' \arrow[dr, "P"] & \\ C \arrow[ur, "S"] \arrow[rr, "F"'] & & D \end{tikzcd}$$

commute. $T\gamma_C$ is a weak equivalence, as a composite of suc. Therefore, S is one as well by the 2-out-of-3 property, so that we may conclude the proof with the commutativity of the latter triangle. $\square$

This concludes the proof of Lemma 2.14.

Observation 2.27. In the factorization above

$$\begin{tikzcd} & X \arrow[dr, "Q"] & \\ LC \arrow[ur, "T"] \arrow[rr, "\bar{F}"'] & & LD, \end{tikzcd}$$

T can be chosen to be a trivial cofibration in the Joyal model structure. Therefore, given another factorization

$$\begin{tikzcd} & X' \arrow[dr, "Q'"] & \\ LC \arrow[ur, "T'"] \arrow[rr, "\bar{F}"'] & & LD \end{tikzcd}$$

of the same form, the commutative square

$$\begin{tikzcd} LC \arrow[r, "T'"] \arrow[d, "T"'] & X' \arrow[d, "Q'"] \\ X \arrow[r, "Q"'] & LD \end{tikzcd}$$

admits, due to the lifting properties of trivial cofibrations with respect to fibrations (or vice versa) in the Joyal model structure, lifts $\ell\colon X \to X'$ and $\ell'\colon X' \to X$, which have to be weak categorical equivalence by the 2-out-of-3 property. This gives us commutative squares

$$\begin{tikzcd} X \arrow[r, "Q"] \arrow[d, "\ell"] & LD \arrow[d, "1_{LD}"] & D \arrow[l, "\gamma_D"'] \arrow[d, "1_D"] \\ X' \arrow[r, "Q'"'] & LD & D \arrow[l, "\gamma_D"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "Q'"] \arrow[d, "\ell'"] & LD \arrow[d, "1_{LD}"] & D \arrow[l, "\gamma_D"'] \arrow[d, "1_D"] \\ X \arrow[r, "Q"'] & LD & D \arrow[l, "\gamma_D"] \end{tikzcd}$$

inducing weak equivalences

$$(\ell, 1_D)\colon X \times_{LD} D \longrightarrow X' \times_{LD} D \quad \text{and} \quad (\ell', 1_D)\colon X' \times_{LD} D \longrightarrow X \times_{LD} D$$

in CofQCat, which are lifts of the commutative square in CofQCat

$$\begin{tikzcd} C \arrow[r, "T'\gamma_C"] \arrow[d, "T\gamma_C"'] & X' \times_{LD} D \arrow[d, "\mathrm{pr}_2"] \\ X \times_{LD} D \arrow[r, "\mathrm{pr}_2"'] & D. \end{tikzcd}$$

Hence, all factorizations of a right exact functor $F\colon C \to D$ in CofQCat which are produced by factorizations of $\bar{F}\colon LC \to LD$ in the Joyal model structure on sSet via pullback are connected by weak equivalences in CofQCat.

Remark. It is unknown to the author whether all factorizations in $\mathrm{CofQCat}_f$ arise in this manner, but, given a factorization in $\mathrm{CofQCat}_f$ of the form

$$\begin{tikzcd} & C' \arrow[dr, "P"] & \\ C \arrow[ur, "S"] \arrow[rr, "F"'] & & D \end{tikzcd}$$

with S a weak equivalence and P a fibration, the resulting commutative square

$$\begin{tikzcd} C' \arrow[r, "P"] \arrow[d, "\gamma_{C'}"'] & D \arrow[d, "\gamma_D"] \\ LC' \arrow[r, "\bar{P}"'] & LD \end{tikzcd}$$

is a homotopy pullback square in $\mathrm{CofQCat}_f$ since the vertical maps are weak equivalences. Therefore, there is a factorization of $\bar{P}$ in $\mathrm{CofQCat}_f$

$$\begin{tikzcd} & X \arrow[dr, "Q"] & \\ LC' \arrow[ur, "T"] \arrow[rr, "\bar{P}"'] & & LD \end{tikzcd}$$

with T a weak equivalence and Q a fibration such that the right exact functor $(T\gamma_C, \gamma_D)\colon C' \to X \times_{LD} D$ induced by the commutative square

$$\begin{tikzcd} C' \arrow[r, "P"] \arrow[d, "T\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ X \arrow[r, "Q"'] & LD \end{tikzcd}$$

is a weak equivalence in CofQCat. In general, all factorizations are connected by a span of weak equivalences since they are isomorphic in the localization, but this result shows us that these weak equivalences can be chosen, so that all domains and codomains form factorizations themselves.

Since this section is originally supposed to be a generalization of [Szu14, 1. Cofibration categories], we would like to recall its construction of path objects in $\mathrm{CofQCat}_f$.

Notation 2.28. Any quasi-category X will be identified with $\mathrm{Fun}\left(\Delta^0, X\right)$ and vice versa. Let C and D be quasi-categories. Then we denote the canonical functor $C \to \Delta^0$ by p_C and the pullback functor $(p_C)^*\colon D \to \mathrm{Fun}(C, D)$ is called *C-ary diagonal functor* of D and denoted by Δ_C. Moreover, for any object c in C, the pullback functor $c^*\colon \mathrm{Fun}(C, D) \to D$ is called *evaluation functor* at c and denoted by ev_c.

Remark. The $\left(\partial\Delta^1\right)$-ary diagonal functor is simply called the (ordinary) diagonal (functor). Note that, for all quasi-categories X, we usually identify $\mathrm{Fun}\left(\partial\Delta^1, X\right)$ with $X \times X$ and vice versa.

Definition 2.29. Let Λ_2^2 denote the $(2,2)$-horn. Then, for any homotopical quasi-category C of cofibrant objects, let $\mathrm{path}(C)$ denote[1] the full subcategory of $\mathrm{Fun}\left(\Lambda_2^2, C\right)$ spanned by spans in C whose legs are trivial cofibrations.

Remark. Objects a in $\mathrm{Fun}\left(\Lambda_2^2, C\right)$ are spans in C of the form

$$a_0 \xrightarrow{i_0} a_2 \xleftarrow{i_1} a_1,$$

and maps $f\colon a \to b$ in $\mathrm{Fun}\left(\Lambda_2^2, C\right)$ are commutative diagrams in C of the form

$$\begin{tikzcd} a_0 \arrow[r, "i_0"] \arrow[d, "f_0"] & a_2 \arrow[d, "f_2"] & a_1 \arrow[l, "i_1"'] \arrow[d, "f_1"] \\ b_0 \arrow[r, "j_0"'] & b_2 & b_1. \arrow[l, "j_1"] \end{tikzcd}$$

Definition 2.30. Let $f\colon a \to b$ be a map in $\mathrm{path}(C)$ of the above form. If f_0 and f_1 are weak equivalences in C, then we call f a weak equivalence. If f_0 and f_1 as well as the induced maps $(j_0, f_2) : b_0 \sqcup_{a_0} a_2 \to b_2$ and $(j_1, f_2) : b_1 \sqcup_{a_1} a_2 \to b_2$ are cofibrations in C, then we call f a cofibration.

Remark. If f as above is a weak equivalence in $\mathrm{path}(C)$, then f_3 is a weak equivalences in C, by the 2-out-of-3 property in C. If f is a cofibration in $\mathrm{path}(C)$, then f_3 is a cofibration in C, as the composite of such. Therefore, weak equivalences and cofibration in $\mathrm{path}(C)$ are componentwise so in C. Note that a map in $\mathrm{path}(C)$ is a weak equivalence if and only if it is componentwise so in C whereas this does not hold for cofibrations.

Proposition 2.31. *[Szu14, Proposition 1.16 & Theorem 1.17 (C5)^{op}]* $\mathrm{path}(C)$ *together with the classes of maps defined above forms a homotopical quasi-category with weak equivalences and cofibrations such that the commutative triangle*

$$\begin{tikzcd} & \mathrm{path}(C) \arrow[dr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & \\ C \arrow[ur, "\Delta_{\Lambda_2^2}"] \arrow[rr, "(1_C, 1_C)"'] & & C \times C \end{tikzcd}$$

forms a path object of C *in* $\mathrm{CofQCat}_f$.

A proof of this is given in the appendix. We would like to sketch how to derive this construction from the methods used in the proof of Lemma 2.26. More precisely, we construct a commutative square that exhibits $\mathrm{path}(C)$ as the homotopy pullback of the canonical path object $\mathrm{Fun}(J, LC)$ of LC.

Observation 2.32. Let C be a homotopical quasi-category of cofibrant objects and denote its localization functor by $\gamma_C\colon C \to LC$. Recall that LC is finitely cocomplete and that γ_C is right exact. One can check[2] that there is an equivalence of ∞-categories of the form $L(C \times C) \to LC \times LC$ which is given by

[1] Although, this has been denoted by $P(C)$ in [Szu14], we choose a different notation because, in the case that C is (the nerve of) a category, $P(C)$ denotes a more general category. See Lemma 4.14.

[2] See [Cis19, Proposition 7.1.13].

the total derived functors of the projections $C \times C \to C$. Under this identification, the diagonal of LC becomes the total derived functor of the diagonal of C. Therefore, any path object of LC produces one of C via pullback.

Recall that, for any quasi-category X, the commutative triangle

$$\begin{tikzcd} & \mathrm{Fun}(J,X) \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & \\ X \arrow[ur, "\Delta_J"] \arrow[rr, "(1_X,1_X)"'] & & X \times X \end{tikzcd}$$

forms a path object of X in the Joyal model structure and, by Lemma 3.7, in CofQCat as well if X is finitely cocomplete. In our case, this means that a pullback in CofQCat of the span

$$\mathrm{Fun}(J,LC) \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} LC \times LC \xleftarrow{\gamma_{C\times C}} C \times C$$

would form a path object of C

$$\begin{tikzcd} & \mathrm{Fun}(J,LC) \times_{LC\times LC} (C \times C) \arrow[dr, "\mathrm{pr}_2"] & \\ C \arrow[ur, "(\Delta_{LC}\gamma_C,(1_C,1_C))"] \arrow[rr, "(1_C,1_C)"'] & & C \times C. \end{tikzcd}$$

Remark. Informally, such a pullback should be a quasi-category of pairs (a,b) of objects in C together with a "witness" of them being isomorphic in LC. If an isomorphism $i\colon a \to b$ in LC is considered to be such a witness in LC, then the span of trivial cofibrations $r\colon a \to x$ and $s\colon b \to x$ that represents i should be such a witness in C.

For the remainder of this subsection, we would like to show that $\mathrm{Fun}(J,LC)$ is a localization of $\mathrm{path}(C)$ and present $\mathrm{Fun}(J,LC)$ by a quasi-category for which there exists a commutative square of the form

$$\begin{tikzcd} \mathrm{path}(C) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[d] & C \times C \arrow[d, "\gamma_{C\times C}"] \\ \mathrm{Fun}(J,LC) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & LC \times LC \end{tikzcd}$$

on the level of simplicial sets.

Lemma 2.33. *We declare all maps in Δ^1 and Λ^2_2 to be weak equivalences, that is, we consider the marked simplicial sets (Δ^1,Δ^1) and $(\Lambda^2_2,\Lambda^2_2)$. Then the total derived functors of the homotopical functors*[1] *$d_1\colon \Delta^1 \to \Lambda^2_2$ and $s_1\colon \Lambda^2_2 \to \Delta^1$. are mutually inverse equivalences of ∞-categories. In particular, the composition of $\overline{s_1}\colon L\Lambda^2_2 \to J$ and $\gamma_{\Lambda^2_2}\colon \Lambda^2_2 \to L\Lambda^2_2$ is a localization of Λ^2_2.*

Remark. We consider J to be the localization of Δ^1 at Δ^1 and would like to present J by a quasi-category such that, for any quasi-category X, the pullback functor $\mathrm{Fun}(J,X) \to \mathrm{Fun}_{\Delta^1}(\Delta^1,X)$ is an isomorphism on the level of simplicial sets. Such a presentation of J can be found in [Cis19, Definition 3.3.3].

[1]These are the degeneracy and face maps and note that they factor or restrict to Λ^2_2, respectively.

In order to prove this assertion, we would like to use the following strategy.

2.34. Let (C, V) and (D, W) be marked quasi-categories and denote their localizations by $\gamma_C : C \to L_V C$ and $\gamma_D : D \to L_W D$, respectively. Moreover, let $\mathrm{Fun}((C, V), (D, W))$ denote the full subcategory of $\mathrm{Fun}(C, D)$ spanned by so-called *homotopical functors*, functors that send V to W. Given such a morphism $F : (C, V) \to (D, W)$, its total derived functor $\bar{F} : L_V C \to L_W D$ is simply the image under the composite

$$\mathrm{Fun}((C,V),(D,W)) \xrightarrow{\gamma_{D,*}} \mathrm{Fun}_V(C, L_W D) \xrightarrow{\gamma_{C,!}} \mathrm{Fun}\,(L_V C, L_W D)$$

where $\gamma_{D,*}$ denotes the functor given by postcomposing γ_D and $\gamma_{C,!}$ is a quasi-inverse to the pullback functor γ_C^*. Hence, any natural transformation $\epsilon : F \Rightarrow G$ between morphisms of marked quasi-categories induces a natural transformation $\bar{\epsilon} : \bar{F} \Rightarrow \bar{G}$ between the total derived functors.

Now, if ϵ belongs componentwise to W, i.e. for any object $c : \Delta^0 \to C$, the evaluation functor $\mathrm{ev}_c : \mathrm{Fun}((C, V), (D, W)) \to D$ sends ϵ to a morphism in W, then $\gamma_{D,*}(\epsilon)$ is componentwise invertible because the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}((C,V),(D,W)) \arrow[r, "\gamma_{D,*}"] \arrow[d, "\mathrm{ev}_c"'] & \mathrm{Fun}_V(C, L_W D) \arrow[d, "\mathrm{ev}_c"] \\
D \arrow[r, "\gamma_D"'] & L_W D
\end{tikzcd}$$

commutes. Therefore[1], $\gamma_{D,*}(\epsilon)$ is a natural isomorphism and so is $\bar{\epsilon} : \bar{F} \Rightarrow \bar{G}$, as the image of $\gamma_{D,*}(\epsilon)$ under $\gamma_{C,!}$.

Proof of lemma. On one hand, the composite $s_1 d_1$ is the identity of Δ^1 and thus the composite $\overline{s_1 d_1}$ is equivalent to the identity of J. On the other hand, it suffices to construct a natural transformation from the identity of Λ^2_2 to the composite $d_1 s_1$ since all maps in Λ^2_2 are weak equivalences. We leave it to the reader to formalize this commutative diagram in Λ^2_2

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2 \arrow[d] & 1 \arrow[l] \arrow[d] \\
0 \arrow[r] & 2 & 2 \arrow[l]
\end{tikzcd}$$

into a functor $\Delta^1 \times \Lambda^2_2 \to \Lambda^2_2$. After transposition, this turns out to be the natural transformation of the desired form. □

Observation 2.35. This diagram in $\mathrm{CofQCat}_f$

$$\begin{tikzcd}
C \arrow[r, "\Delta_{\Lambda^2_2}"] \arrow[d, "\gamma_C"] & \mathrm{path}(C) \arrow[r, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \arrow[d, "\gamma"] & C \times C \arrow[d, "\gamma_{C \times C}"] \\
LC \arrow[r, "\Delta_J"'] & \mathrm{Fun}(J, LC) \arrow[r, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & LC \times LC
\end{tikzcd} \tag{2.35.1}$$

commutes. Here, γ denotes the functor given by postcomposing with γ_C and precomposing with $\overline{d_1}$.

[1] See [Cis19, Corollary 3.5.12].

Proposition 2.36. *Let C be a quasi-category with weak equivalences and cofibrations and denote its localization functor by $\gamma_C \colon C \to LC$. Then the functor $\gamma\colon \operatorname{path}(C) \to \operatorname{Fun}(J, LC)$ is a localization functor.*

Proof. Since all maps in the left inner square of the diagram in (2.35.1) besides the one in question are weak equivalences[1], so is γ by the 2-out-of-3 property. Recall that $\operatorname{Fun}(J, LC)$ is a finitely cocomplete quasi-category, that is, its weak equivalences are the isomorphisms, so that its identity is a localization functor since all functors preserve isomorphisms. Therefore, γ is equivalent to the localization functor of $\operatorname{path}(C)$ and, hence, itself a localization functor. $\square$

Corollary 2.37. *The right inner square of the diagram in (2.35.1) is a homotopy pullback square in* CofQCat*.*

The Homotopy Theory of CofQCat

Definition 2.38. Let $i\colon \mathrm{CofQCat}_f \to \mathrm{CofQCat}$ denote the inclusion functor and let $\ell\colon \mathrm{CofQCat} \to \mathrm{CofQCat}_f$ be the functor that sends any category with weak equivalences and cofibrations C to the saturation of its full subcategory of cofibrant objects. Following the notation of Lemma 2.7 and Lemma 2.8, $\ell(C)$ will be denoted by $\overline{C_c}$ and the canonical weak equivalence in CofQCat are denoted by $\iota_C\colon C_c \to C$ and $\sigma_C\colon C_c \to \overline{C_c}$.

Remark. Since right exact functors preserve cofibrations between cofibrant objects, they preserve cofibrant objects, so that any right exact functor $F\colon C \to D$ restricts to a right exact functor of the form $F_c\colon C_c \to D_c$ which is homotopical by Lemma 1.14. By the construction in paragraph 3.10, there is a commutative diagram of simplicial sets of the form

$$\begin{tikzcd} C_c \arrow[r, "F_c"] \arrow[d, "\gamma_{C_c}"'] & D_c \arrow[d, "\gamma_{D_c}"] \\ LC_c \arrow[r, "\bar{F}_c"'] & LD_c, \end{tikzcd}$$

so that F_c sends $\gamma_{C_c}^{-1}\left((LC_c)^{\simeq}\right)$ to $\gamma_{D_c}^{-1}\left((LD_c)^{\simeq}\right)$. Thus, F_c descends to a right exact functor of the form $\overline{F_c}\colon \overline{C_c} \to \overline{D_c}$. Per construction, this diagram in CofQCat

$$\begin{tikzcd} C \arrow[r, "F"] & D \\ C_c \arrow[u, "\iota_C"] \arrow[r, "F_c"] \arrow[d, "\sigma_C"'] & D_c \arrow[u, "\iota_D"'] \arrow[d, "\sigma_D"] \\ \overline{C_c} \arrow[r, "\overline{F_c}"] & \overline{D_c} \end{tikzcd} \tag{2.38.1}$$

commutes. Using this, one can check that the assignment $C \mapsto \overline{C_c}$ indeed forms a functor ℓ.

[1] A proof of the fact that $\Delta_{\Lambda^2_2}$ is a weak equivalence is given in the Section A, but it should be pointed out that, for the purpose of this proof, it suffices to define a natural transformation between the homotopical functors $\left(p_{\Lambda^2_2}\right)^*$ and ev_0 that is componentwise a weak equivalence. See 2.34.

Proposition 2.39. *$i\colon \mathrm{CofQCat}_f \to \mathrm{CofQCat}$ and $\ell\colon \mathrm{CofQCat} \to \mathrm{CofQCat}_f$ are homotopical functors, whose total derived functors are mutually inverse equivalence $\bar{\imath}\colon L\mathrm{CofQCat}_f \to L\mathrm{CofQCat}$ and $\bar{\ell}\colon L\mathrm{CofQCat} \to L\mathrm{CofQCat}_f$ of ∞-categories.*

Proof. Clearly, i is homotopical. Let $F\colon C \to D$ be a weak equivalence in CofQCat. Since $\iota_C\colon C_c \to C$ and $\sigma_C\colon C_c \to \overline{C_c}$ are weak equivalences in CofQCat, so are $F_c\colon C_c \to D_c$ and thus $\overline{F_c}\colon \overline{C_c} \to \overline{D_c}$ by the 2-out-of-3 property and the commutativity of the diagram in (2.38.1). Therefore, ℓ is homotopical.

Since ℓi is trivially (equivalent to) the identity of $\mathrm{CofQCat}_f$, its total derived functor $\overline{\ell i}$ is equivalent to the identity of $L\mathrm{CofQCat}_f$. By the existence and commutativity of (the upper half of) the diagram in (2.38.1), the assignment $C \mapsto C_c$ forms a functor $\mu\colon \mathrm{CofQCat} \to \mathrm{CofQCat}$. Furthermore, the diagram in (2.38.1) shows that there are two natural transformations $\iota\colon \mu \Rightarrow \mathrm{id}_{\mathrm{CofQCat}}$ and $\sigma\colon \mu \Rightarrow i\ell$, which are componentwise the weak equivalences $\iota_C\colon C_c \to C$ and $\sigma_C\colon C_c \to \overline{C_c}$, respectively. By Lemma 2.34, this implies that the induced natural transformations $\bar{\iota}\colon \bar{\mu} \Rightarrow \mathrm{id}_{L\mathrm{CofQCat}}$ and $\bar{\sigma}\colon \bar{\mu} \Rightarrow \overline{i\ell}$ are natural isomorphisms. Hence, $\overline{i\ell}$ is equivalent to $\mathrm{id}_{L\mathrm{CofQCat}}$. □

The Pseudo-factorization Axiom

Although CofQCat lacks the factorization axiom, it admits a weaker version of it: the so-called *pseudo-factorization axiom.* In the following, we will show that it implies statements, analogous to the factorization lemma and Ken Brown's lemma.

Proposition 2.40. *Any map in $F\colon C \to D$ in* CofQCat *admits a pseudo-factorization.*

Proof. By a previous remark, there is a commutative square in CofQCat of the form

$$\begin{tikzcd} C_c \arrow[r, "F_c"] \arrow[d, "\iota_C"'] & D_c \arrow[d, "\iota_D"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

where ι_C and ι_D are weak equivalence by Lemma 2.7, and it is left to the reader to show that they are fibrations in CofQCat as well. Moreover, note that Lemma 2.25 holds even in the case that the classes of weak equivalences are not saturated or, equivalently, do not satisfy the 2-out-of-of-6 property. Therefore, the proof of Lemma 2.26 is still applicable to $F_c\colon C_c \to D_c$, that is, any factorization of $\bar{F}_c\colon LC_c \to LD_c$ produces a factorization of $F_c\colon C_c \to D_c$ via pullback. Let

$$\begin{tikzcd} & E \arrow[dr, "P"] & \\ C_c \arrow[ur, "S"] \arrow[rr, "F_c"'] & & D_c \end{tikzcd}$$

denote such a factorization in CofQCat. Then

$$
\begin{tikzcd}
C_c \arrow[r, "S"] \arrow[d, "\iota_C"'] & E \arrow[d, "\iota_D P"] \\
C \arrow[r, "F"'] & D
\end{tikzcd}
$$

forms a pseudo-factorization of F in CofQCat. $\square$

Remark. The obstacle that prevents us from constructing factorizations for all right exact functors in CofQCat with fibrant codomain is the fact that right exact functors with arbitrary domain do not preserve all weak equivalences, but only those between cofibrant objects. A possible solution would be the ability to construct cofibrant replacements that form right exact functors $C \to C_c$.

Proposition 2.41 (Pseudo-factorization Lemma)**.** *Let $F\colon C \to D$ be a morphism in* CofQCat*. Then there exists a commutative diagram of the form*

$$
\begin{tikzcd}
C \arrow[d, "1_C"] & X \arrow[l, "T"'] \arrow[r, "T"] \arrow[d, "S"] & C \arrow[d, "F"] \\
C & Y \arrow[l, "Q"] \arrow[r, "P"'] & D
\end{tikzcd}
\tag{2.41.1}
$$

where S is a weak equivalence, P is a fibration and Q and T are trivial fibrations in CofQCat*.*

Proof. Analogous to the proof of Lemma 1.13, this follows from a pseudo-factorization of the morphism $(1_C, F) : C \to C \times D$. $\square$

Corollary 2.42. *Let D be a quasi-category and let V be a subcategory satisfying the 2-out-of-3 property. Then any functor $\mathcal{F}\colon$ CofQCat $\to D$ that sends trivial fibrations to maps in V sends all weak equivalences to maps in V.*

Proof. Let $F\colon C \to D$ be a weak equivalence in CofQCat and choose a commutative diagram of the form in (2.41.1). Analogous to the proof Lemma 1.14, use the 2-out-of-3 property to observe that S is sent to a map in V since T and Q are sent to maps in V. Moreover, P is a trivial fibration and thus sent to a map in V. Therefore, the composite FT is sent to a map in V and, since T is already sent to a map in V, so is F. $\square$

3 The Homotopy Theory of Finitely Cocomplete Quasi-categories

Definition 3.1. Let CofCat denote the category defined by the following data:

- An object is a $\mathbf{U}$-small category C together with a structure of a quasi-category with weak equivalences and cofibrations on NC.
- Given two objects C and D, a morphism is a functor $F\colon C \to D$ such that $NF\colon NC \to ND$ is right exact.
- The composition law is given by the ordinary composition law of functors between categories.

The objects and morphisms will be called *categories with weak equivalences and cofibrations* and *right exact functors*, respectively. Furthermore, CofCat is equipped with two classes of morphisms called *fibrations* and *weak equivalences* which are defined to be those right exact functors $F\colon C \to D$ such that $NF\colon NC \to ND$ are fibrations and weak equivalences of quasi-categories with weak equivalences and cofibrations, respectively. Analogous to the definition of $\mathrm{CofQCat}_f$, we define the subcategory CofCat_f. By an abuse of notation, we denote the functor $\mathrm{CofCat}_f \to \mathrm{CofQCat}_f$ induced by sending categories to their nerves by N as well.

Remark. If $\mathbf{V}$ is a Grothendieck universe for which Cat, CofQCat and sSet are small, CofCat is the pullback of the span of $\mathbf{V}$-small categories

$$\mathrm{Cat} \xrightarrow{N} \mathrm{sSet} \xleftarrow{\mathbf{forget}} \mathrm{CofQCat}$$

where **forget** denotes the forgetful functor that sends quasi-categories with weak equivalences and cofibrations to their underlying quasi-categories. Similarly, the subcategories of fibrations and weak equivalences and CofCat_f are pullbacks of the respective subcategories of CofQCat.

Proposition 3.2. CofCat_f *is a category with weak equivalences and fibrations such that N is left exact.*

Proof. CofCat admits a terminal object, namely $[0]$ with the unique structure where the identity of 0 is a trivial cofibration: As $[0]$ is the terminal category, for any category with weak equivalences and cofibrations C, there is unique functor $C \to [0]$, which sends any map in C to the identity of 0. In particular, $C \to [0]$ is right exact. Moreover, N sends $[0]$ to Δ^0. Due to the functoriality of the nerve construction, the class of weak equivalences in CofCat is closed under identities and satisfies the 2-out-of-3 property, and the class of fibrations in CofCat contains all identities and is closed under composition. The pullback axioms for fibrations and trivial fibrations hold due to the fact that N commutes with limits, as N admits a left adjoint. In particular, this shows that N is left exact. Lastly, the factorization axiom follows from the fact that, given a category with weak equivalences and cofibrations C, the path object $\mathrm{path}(NC)$ in CofQCat as in Lemma 2.31 is the nerve of a category. $\square$

Definition 3.3. Let RexQCat be the subcategory of sSet spanned by **U**-small quasi-categories with finite colimits and finite colimit preserving functors between them. There are various different names for the objects and morphisms in RexQCat, but we choose to call them *finite cocomplete* quasi-categories and *finitely cocontinuous* functors, respectively.

3.4. Finitely cocomplete quasi-categories carry a canonical structure of homotopical quasi-categories of cofibrant objects where all maps are cofibrations and only the isomorphisms are weak equivalences. With respect to this structure, the right exact functors between finitely cocomplete quasi-categories are precisely the finitely cocontinuous ones. Therefore, the assignment $X \mapsto (X, X^{\simeq}, X)$ forms a fully faithful functor $i' \colon \mathrm{RexQCat} \to \mathrm{CofQCat}_f$.

Definition 3.5. A morphism in RexQCat is called weak equivalence or fibration if i' sends it to a weak equivalence or fibration in $\mathrm{CofQCat}_f$, respectively.

Observation 3.6. Since functors between quasi-categories preserve isomorphisms, the identity of a quasi-category X is canonically a localization functor of X at its isomorphisms. Therefore, weak equivalences of finitely cocomplete quasi-categories are precisely the weak categorical equivalences, that is, weak equivalences in the Joyal model structure of simplicial sets. Moreover, fibrations of finitely cocomplete quasi-categories are precisely those right exact functors that are inner fibrations and isofibrations, that is, fibrations in the Joyal model structure, as the property of lifting factorizations and pseudo-factorizations follow immediately from the property of being an isofibration since the weak equivalences are the isomorphisms and all maps are cofibrations.

Informally, RexQCat lies in the intersection of the Joyal model structure of simplicial sets and the right exact structure of $\mathrm{CofQCat}_f$ and we will exploit this fact in the following lemma.

Lemma 3.7. RexQCat *together with the classes of fibrations and weak equivalences as above forms a homotopical category of fibrant objects.*

Proof of lemma. We leave these simple observation to the reader:

- RexQCat admits a terminal object, namely Δ^0.
- The class of fibrations in RexQCat contains all identities and is closed under composition.
- The class of weak equivalences in RexQCat is closed under identities and satisfies the 2-out-of-3 property.

Additionally, for any finitely cocomplete quasi-category X, the canonical map $X \to \Delta^0$ is an inner fibration and an isofibration since quasi-categories are the fibrant objects in the Joyal model structure and functors that are inner fibrations and isofibrations are its fibrations between fibrant objects. Moreover, the weak equivalences in RexQCat satisfies the 2-out-of-6 property as they belong to the class of weak equivalence in the Joyal model structure and model categories are saturated.

The proof of the pullback axioms for fibrations and trivial fibrations in RexQCat is analogous to that of Lemma 2.24 as it is mainly an application of [Cis19, Proposition 6.2.19]. Alternatively, given a span in RexQCat of the form

$$X \xrightarrow{F} Y \xleftarrow{P} Z$$

with P a (trivial) fibration, one could form the pullback in $\mathrm{CofQCat}_f$

$$\begin{tikzcd} X \sqcup_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "P"] \\ X \arrow[r, "F"'] & Y. \end{tikzcd}$$

Per construction of Lemma 2.23, the underlying simplicial sets form a pullback square in sSet. Since P is a (trivial) fibration in the Joyal model structure by the preceding observation, so is π_1. By [Cis19, Proposition 6.2.19], one can check that $X \times_Y X$ is finitely cocomplete and the projections are finitely cocontinuous, so that π_1 is a (trivial) fibration in RexQCat and the commutative square is a pullback square in RexQCat as well.

For the factorization axiom, let $F\colon X \to Y$ be a finitely cocontinuous functor between finitely cocomplete quasi-categories and let

$$\begin{tikzcd} & X' \arrow[dr, "P"] & \\ X \arrow[ur, "S"] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

be a factorization in the Joyal model structure, i.e. S is a weak categorical equivalence and P is an inner fibration and an isofibration. Note that X and X' are quasi-categories, so that S presents an equivalence of ∞-categories, which preserve all colimits, in particular those of interest. As mentioned in the preceding observation, S is a weak eqivalence in RexQCat and P satisfies the defining lifting properties for fibrations in RexQCat. Therefore, it remains to show that P is finitely cocontinuous, but this follows from the fact that S reflects all colimits as well and that F is finitely cocontinuous. $\square$

Corollary 3.8. *Let* $\lambda\colon \mathrm{RexQCat} \to \mathrm{RexCat}_\infty$ *denote the localization functor of* RexQCat. *Then* $\mathrm{RexQCat}_\infty$ *is finitely complete and* λ *is finitely continuous.*

Definition 3.9. Let $L'\colon \mathrm{CofQCat}_f \to \mathrm{RexQCat}$ be the functor given by forming localizations and total right derived functors and define $L\colon \mathrm{CofCat}_f \to \mathrm{RexQCat}$ to be the composition of L' and $N\colon \mathrm{CofCat}_f \to \mathrm{CofQCat}_f$.

Remark. Localizations of small quasi-categories (at small classes of maps) are small again.

3.10. [Cis19, Proposition 7.1.3] The localization of a simplicial set C by a simplicial subset W can be constructed in the following way:

1. Choose a fibrant replacement $i\colon W \to W'$ in the Kan-Quillen model structure of simplicial sets.
2. Form the pushout square in sSet:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "i"'] & C \arrow[d, "j"] \\ W' \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C'. \end{tikzcd}$$

3. Choose a fibrant replacement $k\colon C' \to LC$ in the Joyal model structure of simplicial sets.

Then $\gamma_C := kj$ is a localization of C by W. Note that each step in the construction is functorial. In particular, for any simplicial set D with subcategory V and any functor $F\colon C \to D$ sending W to V, we get a commutative diagram in sSet of the form

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, "\bar{F}"'] & LD \end{tikzcd} \tag{3.10.1}$$

exhibiting $\bar{F}$ as a total derived functor of F.

Remark. In the context of Kan extensions, total derived functors $\bar{F}\colon LC \to LD$ may be thought of as the "best solution" to the filling problem

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, dashed] & LD \end{tikzcd}$$

in that $\gamma_C^*(\bar{F})$ is the "closest approximation" to $\gamma_D F$. In the ∞-categorical viewpoint, this means that $\gamma_C^*(\bar{F})$ and $\gamma_D F$ are equivalent at best, but the above construction shows that, in case of a homotopical functor F, the localizations LC and LD can be modelled in a way such that the filling problem can be solved in the strictest sense possible, that is $\gamma_C^*(\bar{F})$ and $\gamma_D F$ are equal as functors of simplicial sets.

Therefore, it might not be sensible to talk about total left derived functors in case of right exact functors between homotopical quasi-categories of cofibrant objects as this terminology has the connotation of a best approximated, rather than a strict solution to a filling problem. Nevertheless, since morphisms in CofQCat are not homotopical, and since $\mathrm{CofQCat}_f$ is treated as a subcategory of it, we choose to use the terminology of total derived functors.

Proposition 3.11. *The total derived functors of $L'\colon \mathrm{CofQCat}_f \to \mathrm{RexQCat}$ and $i'\colon \mathrm{RexQCat} \to \mathrm{CofQCat}_f$ are equivalences of ∞-category that are mutually inverse to each other.*

Proof. By the commutativity of the diagram in (3.10.1), the specific localization functors construced in 3.10 define natural transformations $\mathrm{id}_{\mathrm{CofQCat}_f} \Rightarrow i'L'$ and $\mathrm{id}_{\mathrm{RexQCat}} \Rightarrow L'i'$, which are componentwise weak equivalences. By Lemma 2.34, these induce natural isomorphisms $\mathrm{id}_{L\mathrm{CofQCat}} \Rightarrow \bar{i}'\bar{L}'$ and $\mathrm{id}_{\mathrm{RexCat}_\infty} \Rightarrow \bar{L}'\bar{i}'$. □

Since $L\colon \mathrm{CofCat}_f \to \mathrm{RexQCat}$ preserves all weak equivalences, there is a total derived functor $\bar{L}\colon L\mathrm{CofCat}_f \to \mathrm{RexCat}_\infty$, which we wish to exhibit as an equivalence of ∞-categories. Instead of doing so directly, we would like to make use of this black box:

Theorem 3.12. *[Cis19, Theorem 7.6.10] Let X and Y be two finitely complete ∞-categories. Then any finitely continuous functor $F\colon X \to Y$ is an equivalence of ∞-categories if and only if it satisfies the right approximation property:*

1. F is conservative.

2. For any map $g\colon x \to F(b)$ in D with b an object in C, there is a commutative triangle in D of the form

$$\begin{tikzcd} & F(a) \arrow[dr, "F(f)"] & \\ x \arrow[ur, "i"] \arrow[rr, "g"'] & & F(b) \end{tikzcd}$$

with i an isomorphism in D and $f\colon a \to b$ a map in C.

For this, $\bar{L}$ needs to be finitely continuous. By Lemma 2.11, this is the case when $\bar{L}$ is the total right derived functor of a left exact functor of quasi-categories with weak equivalences and fibrations. As RexCat_∞ is finitely complete, it carries a canonical structure of a quasi-category with weak equivalences and fibrations, so that its identity forms a localization functor and $\bar{L}$ becomes a total derived functor for $\lambda L\colon \mathrm{CofCat}_f \to \mathrm{RexCat}_\infty$ with respect to this structure. For this reason, we show the following:

Corollary 3.13. *The composite $\lambda L\colon \mathrm{CofCat}_f \to \mathrm{RexCat}_\infty$ is a left exact functor.*

Proof. By the preceding proposition, there is a commutative diagram of the form

$$\begin{tikzcd} \mathrm{CofQCat}_f \arrow[r, "L'"] \arrow[d, "\gamma_{\mathrm{CofQCat}_f}"'] & \mathrm{RexQCat} \arrow[d, "\lambda"] \\ L\mathrm{CofQCat}_f \arrow[r, "\bar{L}'"'] & \mathrm{RexCat}_\infty \end{tikzcd}$$

with $\bar{L}'$ an equivalence of ∞-categories. Since L' is homotopical, such a commutative diagram exists on the level of simplicial sets. Therefore, $\lambda L'$ is left exact, as a composition of the left exact functor $\bar{L}'$ and $\gamma_{\mathrm{CofQCat}_f}$. Moreover, $N\colon \mathrm{CofCat}_f \to \mathrm{CofQCat}_f$ is clearly left exact. Thus λL is left exact, as the composition of $\lambda L'$ and N. □

4 A Rectification of Right Exact Functors

In our pursuit of showing that $\bar{L}\colon L\mathrm{CofCat}_f \to \mathrm{RexCat}_\infty$ is an equivalence of ∞-categories, we have done all the preparatory work to be able to conclude with the verification of the right approximation property for $\bar{L}$. Per definition of weak equivalences in $\mathrm{CofQCat}_f$, the functor $L\colon \mathrm{CofCat}_f \to \mathrm{RexQCat}$ reflects weak equivalences, and, because of the 2-out-of-6 property, the class of weak equivalences of both CofCat_f and RexQCat are saturated. Therefore, $\bar{L}$ is conservative, and it remains to show that, for any finitely cocontinuous functor $\mathcal{F}\colon X \to LD$, there is a commutative triangle in RexCat_∞ of the form

$$\begin{tikzcd} X \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{E}"'] & LD \\ LC \arrow[ur, "\bar{L}(F)"'] & \end{tikzcd}$$

where $\mathcal{E}$ is an equivalence of ∞-categories and $F\colon C \to D$ is a map in $L\mathrm{CofCat}_f$. We will refer to the problem of finding such a commutative triangle as the *rectification problem* for finitely cocontinuous functors.

A Functorial and Right Exact Presentation

As the first step in solving this problem, we will functorially exhibit X as the localization of a homotopical category of cofibrant objects. For this, we would like to use existent presentations of ∞-categories by ordinary categories. In our case, we make use of the *contravariant model structure*[1] over X on sSet/X presenting $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$, but the following arguments work for any presentation satisfying the conditions listed in Lemma 4.9.

This model category will be denoted by $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ and the *Grothendieck construction for ∞-groupoids*[2] constitutes an equivalence of the localization of $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ and $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$, that is, there is a localization functor of the form $(\mathrm{sSet}/X)_{\mathrm{rfib}} \to \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$, which we will denote by ρ_X. The particular reason for the choosing $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is that it is a homotopical category of cofibrant objects since its cofibrations are defined to be the monomorphisms. By our computations in Section 1, this implies that ρ_X is right exact, and we wish the same to hold for the Yoneda embedding $よ_X\colon X \to \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ via which X has been embedded into $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$, but the following minimal example shows that this is not the case in general.

Example 4.1. The canonically commutative diagram in Δ^1

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

[1] For the existence and a (partial) definition, see [Cis19, Theorem 4.1.5].
[2] An account of this is given in [Cis19, Theorem 7.8.9]

is a pushout square, but the corresponding commutative square in $\mathrm{Fun}\left(\Delta^1, \mathcal{S}\right)$

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\Delta^1}(-,0) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\Delta^1}(-,1) \arrow[d] \\ \underline{\mathrm{Hom}}_{\Delta^1}(-,1) \arrow[r] & \underline{\mathrm{Hom}}_{\Delta^1}(-,1) \end{tikzcd}$$

is not because, at component 1, the induced commutative square in $\mathcal{S}$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{1_1\} \arrow[d] \\ \{1_1\} \arrow[r] & \{1_1\} \end{tikzcd}$$

is not a pushout square. Here, $\{1_1\}$ denotes Δ^0 where the unique object is interpreted as the identity of 1 in Δ^1. Then one can see that the pushout should be $\Delta^0 \sqcup \Delta^0$, which obviously cannot be equivalent to Δ^0.

The conceptual idea of turning よ_X right exact, i.e. finitely cocontinuous, is to consider an ∞-category of presheaves on X for which the presheaf represented by the initial object is equivalent to the initial presheaf and for which pushouts of representable presheaves are representable up to equivalence. This is formalized as follows.

Definition 4.2. Let $\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ be the localization of $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ at the following natural transformations:

- $\varnothing_{\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S})} \to \text{よ}_X\,(\varnothing_X)$;
- $\text{よ}_X(x) \sqcup_{\text{よ}_X(y)} \text{よ}_X(z) \to \text{よ}_X(x \sqcup_y z)$ for any cospan $x \leftarrow y \to z$ in X.

As X is $\mathbf{U}$-small, so is this class of natural transformations, so that, by [Cis19, Proposition 7.11.4], the localization forms a left Bousfield localization of the form

$$\begin{tikzcd} \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left, "\ell_X"] \arrow[r, phantom, "\perp"] & \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}) \arrow[l, shift left, "\iota_X"] \end{tikzcd}$$

where the left adjoint ℓ_X is the localization functor and the right adjoint ι_X is fully faithful. In particular, $\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ is presentable and, by [Cis19, Proposition 7.11.7], complete and cocomplete. Additionally, ι_X induces an equivalence of $\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ and the full subcategory of $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ spanned by the finitely *continuous* presheaves, which よ_X factors through since representable presheaves are (finitely) cocontinuous. Therefore, the triangle of ∞-categories

$$\begin{tikzcd} & \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}) \arrow[dr, "\iota_X"] & \\ X \arrow[ur, "\ell_X \text{よ}_X"] \arrow[rr, "\text{よ}_X"'] & & \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S}) \end{tikzcd}$$

commutes (up to homotopy[1]).

[1] See Lemma 4.10

Remark. Although $\text{よ}_X \colon X \to \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ is the universal fully faithful embedding of X into a ($\mathbf{U}$-small) cocomplete ∞-category[1], the (existent) colimits of X and $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ are usually independent of each other in the sense that よ_X does not preserve them. That is the reason why よ_X is called the *free* cocompletion of X. In this context, the above definition presents a construction of universally and fully faithfully embedding X into a cocomplete ∞-category in a way that preserves the existing colimits of X in its cocompletion:

Given a simplicial set I and an I-cocomplete ∞-category C, let $\mathrm{Fun}_I\,(C^{\mathrm{op}}, \mathcal{S})$ be the full subcategory of $\mathrm{Fun}\,(C^{\mathrm{op}}, \mathcal{S})$ spanned by the I-continuous presheaves. Because representable presheaves are (I-)continuous, the Yoneda embedding $\text{よ}_C \colon C \to \mathrm{Fun}\,(C^{\mathrm{op}}, \mathcal{S})$ factors through $\mathrm{Fun}_I\,(C^{\mathrm{op}}, \mathcal{S})$, and one can show that $\text{よ}_C \colon C \to \mathrm{Fun}_I\,(C^{\mathrm{op}}, \mathcal{S})$ is the universal I-cocontinuous fully faithful embedding of X into a cocomplete ∞-category.

Lemma 4.3. *Let X be a finitely cocomplete ∞-category. Then the above defined functor $\ell_X \text{よ}_X \colon X \to \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ is fully faithful and finitely cocontinuous. Moreover, it is universal among the finitely cocontinuous functors of the form $X \to D$ with D a ($\mathbf{U}$-small) cocomplete ∞-category in the sense that precomposition with $\ell_X \text{よ}_X$ induces an equivalence of ∞-categories*

$$\left(\ell_X \text{よ}_X\right)^* \colon \mathrm{Fun}_!\,(\mathrm{Lex}(X^{\mathrm{op}}, \mathcal{S}), D) \longrightarrow \mathrm{Rex}\,(X, D)\,.$$

Here, $\mathrm{Fun}_!\,(\mathrm{Lex}(X^{\mathrm{op}}, \mathcal{S}), D)$ *denotes the full subcategory of* $\mathrm{Fun}\,(\mathrm{Lex}(X^{\mathrm{op}}, \mathcal{S}), D)$ *spanned by the cocontinuous functors, and* $\mathrm{Rex}\,(X, D)$ *denotes the full subcategory of* $\mathrm{Fun}\,(X, D)$ *spanned by the right exact, i.e. finitely cocontinuous, functors.*

Proof. Per definition of $\ell_X \colon \mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$, the composite functor $\ell_X \text{よ}_X$ is finitely cocontinuous. Because $\iota_X \ell_X \text{よ}_X$ and よ_X are equivalent, for any pair of objects x and y in X, the diagram of mapping spaces

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathrm{Lex}(X^{\mathrm{op}}, \mathcal{S})}\left(\ell_X \text{よ}_X(x), \ell_X \text{よ}_X(y)\right) \arrow[dr, "\iota_X"] & \\
\mathrm{Map}_X(x, y) \arrow[ur, "\ell_X \text{よ}_X"] \arrow[rr, "\text{よ}_X"] & & \mathrm{Map}_{\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S})}\left(\text{よ}_X(x), \text{よ}_X(y)\right)
\end{tikzcd}$$

commutes (up to equivalence). As よ_X and ι_X are fully faithful, so is $\ell_X \text{よ}_X$ by the 2-out-of-3 property for weak homotopy equivalences.

Let S denote the ($\mathbf{U}$-small) class of natural transformations in Lemma 4.2, and let D be a finitely cocomplete ∞-category. By [Cis19, Remark 7.7.10],

$$\left(\ell_X\right)^* \colon \mathrm{Fun}\,(\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})\,, D) \longrightarrow \mathrm{Fun}\,(\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})\,, D)$$

induces an equivalence of $\mathrm{Fun}_!\,(\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})\,, D)$ and $\mathrm{Fun}_{!,S}\,(\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})\,, D)$, the full subcategory of $\mathrm{Fun}\,(\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})\,, D)$ spanned by cocontinuous functors which send elements of S to isomorphisms in D. Now, we would like to show that the essential image of $\mathrm{Fun}_{!,S}\,(\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})\,, D)$ under

$$\left(\text{よ}_X\right)^* \colon \mathrm{Fun}\,(\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})\,, D) \longrightarrow \mathrm{Fun}\,(X, D)$$

[1]See [Cis19, Theorem 6.3.13].

is $\mathrm{Rex}(X, D)$. So, let $\mathcal{F}\colon \mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right) \to D$ be a cocontinuous functor sending elements of S to isomorphisms in D. Then $\mathcal{F}\text{よ}_X\colon X \to D$ is finitely cocontinuous due to these observations:

- Since $\mathcal{F}$ is cocontinuous, $\mathcal{F}\left(\varnothing_{\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S})}\right)$ is initial in D, so that the canonical map $\varnothing_D \to \mathcal{F}\left(\text{よ}_X\left(\varnothing_X\right)\right)$ is (isomorphic to) the map which $\mathcal{F}$ sends $\varnothing_{\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S})} \to \text{よ}_X\left(\varnothing_X\right)$ to. As $\mathcal{F}$ sends elements of S to isomorphisms in D, the map $\varnothing_D \to \mathcal{F}\left(\text{よ}_X\left(\varnothing_X\right)\right)$ is an isomorphism, i.e. $\mathcal{F}\left(\text{よ}_X\left(\varnothing_X\right)\right)$ is initial in D.

- For the same reason,
$$\mathcal{F}\left(\text{よ}_X(x) \sqcup_{\text{よ}_X(y)} \text{よ}_X(z)\right) \longrightarrow \mathcal{F}\left(\text{よ}_X\left(x \sqcup_y z\right)\right)$$
is an isomorphism. Due to cocontinuity of $\mathcal{F}$, the canonical map
$$\mathcal{F}\left(\text{よ}_X(x)\right) \sqcup_{\mathcal{F}\left(\text{よ}_X(y)\right)} \mathcal{F}\left(\text{よ}_X(z)\right) \longrightarrow \mathcal{F}\left(\text{よ}_X(x) \sqcup_{\text{よ}_X(y)} \text{よ}_X(z)\right)$$
is an isomorphism, so that, by composing these two isomorphisms, one can see that $\mathcal{F}\left(\text{よ}_X\left(x \sqcup_y z\right)\right)$ is isomorphic to $\mathcal{F}\left(\text{よ}_X(x)\right) \sqcup_{\mathcal{F}\left(\text{よ}_X(y)\right)} \mathcal{F}\left(\text{よ}_X(z)\right)$.

Conversely, let $F\colon X \to D$ be a finitely cocontinuous functor. Then we would like to show that there is a cocontinuous functor $\mathcal{F}\colon \mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right) \to D$ which sends elements of S to isomorphisms in D such that F is isomorphic to $\mathcal{F}\text{よ}_X$. By [Cis19, Theorem 6.3.13], precomposition with よ_X induces an equivalence of $\mathrm{Fun}_!\left(X^{\mathrm{op}}, D\right)$, the full subcategory of $\mathrm{Fun}\left(X^{\mathrm{op}}, D\right)$ spanned by cocontinuous presheaves, and $\mathrm{Fun}(X, D)$, so that there exists a cocontinuous functor $\mathcal{F}\colon \mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right) \to D$ such that $F \cong \mathcal{F}\text{よ}_X$ and we are left to show that $\mathcal{F}$ sends elements of S to isomorphisms in D. Note that $\mathrm{Rex}(X, D)$ is a replete subcategory of $\mathrm{Fun}(X, D)$, which means that any functor $X \to D$ that is isomorphic to a finitely cocontinuous one is already finitely cocontinuous.

- As $\mathcal{F}$ is cocontinuous, $\mathcal{F}\left(\varnothing_{\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S})}\right)$ is initial in D, and, as $\mathcal{F}\text{よ}_X$ is finitely cocontinuous, $\mathcal{F}\left(\text{よ}_X\left(\varnothing_X\right)\right)$ is initial as well. Hence, the identity of $\varnothing_D$ is isomorphic to the image $\varnothing_{\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S})} \to \text{よ}_X\left(\varnothing_X\right)$ under $\mathcal{F}$.

- Analogously, for any cospan $x \leftarrow y \to z$ in X,
$$\mathcal{F}\left(\text{よ}_X(x)\right) \sqcup_{\mathcal{F}\left(\text{よ}_X(y)\right)} \mathcal{F}\left(\text{よ}_X(z)\right) \longrightarrow \mathcal{F}\left(\text{よ}_X(x) \sqcup_{\text{よ}_X(y)} \text{よ}_X(z)\right)$$
is an isomorphism because $\mathcal{F}$ is cocontinuous, and
$$\mathcal{F}\left(\text{よ}_X(x)\right) \sqcup_{\mathcal{F}\left(\text{よ}_X(y)\right)} \mathcal{F}\left(\text{よ}_X(z)\right) \longrightarrow \mathcal{F}\left(\text{よ}_X\left(x \sqcup_y z\right)\right)$$
is an isomorphism because $\mathcal{F}\text{よ}_X$ is finitely cocontinuous. Then the image of $\text{よ}_X(x) \sqcup_{\text{よ}_X(y)} \text{よ}_X(z) \to \text{よ}_X(x \sqcup_y z)$ under $\mathcal{F}$ is the composition of isomorphisms.

Thus, $\mathcal{F}$ sends elements of S to isomorphisms in D, and this shows that $\left(\text{よ}_X\right)^*$ factors through $\mathrm{Fun}_{!,S}\left(\mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right), D\right) \to \mathrm{Rex}(X, D)$, which is essentially surjective. As a restriction of the equivalence $\mathrm{Fun}_!\left(X^{\mathrm{op}}, D\right) \to \mathrm{Fun}(X, D)$ of ∞-categories, it is fully faithful, thus itself an equivalence of ∞-categories. Therefore, the composite
$$\mathrm{Fun}_!\left(\mathrm{Lex}\left(X^{\mathrm{op}}, \mathcal{S}\right), D\right) \xrightarrow{\left(\ell_X\right)^*} \mathrm{Fun}_{!,S}\left(\mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right), D\right) \xrightarrow{\left(\text{よ}_X\right)^*} \mathrm{Rex}\left(X, D\right)$$

is an equivalence of ∞-categories. $\square$

Definition 4.4. Let S denote the preimage of $(\mathrm{sSet}/X)^{\simeq}_{\mathrm{lex}}$ under $\ell_X \rho_X$, i.e. the pullback of the span

$$\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})^{\simeq} \hookrightarrow \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}) \xleftarrow{\ell_X \rho_X} (\mathrm{sSet}/X)_{\mathrm{rfib}}.$$

Define $(\mathrm{sSet}/X)_{\mathrm{lex}}$ to be the left Bousfield localization of $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ at S and denote by

$$\begin{tikzcd} (\mathrm{sSet}/X)_{\mathrm{rfib}} \arrow[r, shift left, "\ell_X^\flat"] \arrow[r, phantom, "\perp"] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[l, shift left, "\iota_X^\flat"] \end{tikzcd}$$

the associated Quillen adjunction given by the identity of the underlying category sSet/X.

Since the class of cofibrations in $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ and $(\mathrm{sSet}/X)_{\mathrm{lex}}$ are the same, all objects in $(\mathrm{sSet}/S)_{\mathrm{lex}}$ are cofibrant, so that $(\mathrm{sSet}/S)_{\mathrm{lex}}$ is a homotopical category of cofibrant objects.

Remark. Note that neither $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ nor $\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ are $\mathbf{U}$-small, so that S cannot be expected to be $\mathbf{U}$-small either. Therefore, [Cis19, Proposition 7.11.4] is not applicable in order to ensure the existence of $(\mathrm{sSet}/X)_{\mathrm{lex}}$. Nevertheless, if we choose a Grothendieck universe $\mathbf{V}$ for which all of the above are $\mathbf{V}$-small, $(\mathrm{sSet}/X)_{\mathrm{lex}}$ can be shown to exist as follows:

$(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is a cofibrantly generated model category by [Cis19, Theorem 4.1.5 & Theorem 2.4.19] and $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is locally presentable because of $\mathrm{sSet}/X \cong \mathrm{Fun}\,((\Delta/X)^{\mathrm{op}}, \mathrm{Set})$, thus a combinatorial model category. As the cofibrations are precisely the monomorphisms, all objects in $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ are cofibrant, so that $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is left proper. By [Lur09, Proposition 2.1.4.8 & Remark 2.1.4.12], $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is a simplicial model category. For this reason, [Bar10, Theorem 4.7] implies the existence of $(\mathrm{sSet}/X)_{\mathrm{lex}}$.

Observation 4.5. By [Cis19, Remark 7.7.10], the total derived functor of $\ell_X^\flat$ is a localization functor of $\mathrm{Fun}\,(X^{\mathrm{op}}, \mathcal{S})$ at the image of S under ρ_X. Note that ℓ_X is defined to be such a localization functor. Since the formation of total derived functors preserve homotopical adjunctions[1] and since right (or left) adjoint functors are unique up to equivalence[2], the total derived functor of the right adjoint $\iota_X^\flat$ is right adjoint to ℓ_X and thus equivalent to ι_X. In particular, there is a localization functor of the form $\rho_{X,\mathrm{lex}} \colon (\mathrm{sSet}/X)_{\mathrm{lex}} \to \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$.

Definition 4.6. Let $X^\flat$ be defined to be the homotopical quasi-category of cofibrant objects such that there is a homotopy pullback square of the form

$$\begin{tikzcd} X^\flat \arrow[r] \arrow[d, "\varrho_X"'] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[d, "\rho_{X,\mathrm{lex}}"] \\ X \arrow[r, "\ell_X \text{よ}_X"'] & \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}). \end{tikzcd}$$

Since $\ell_X \text{よ}_X$ is fully faithful by the preceding lemma, $X^\flat$ is equivalent to a full subcategory of $(\mathrm{sSet}/X)_{\mathrm{lex}}$.

[1] See [Cis19, Proposition 7.1.14].
[2] See [Cis19, Proposition 6.1.9].

Proposition 4.7. *If X is a* $\mathbf{U}$*-small finitely cocomplete quasi-category, then $X^\flat$ as defined above is weakly equivalent to an object in* CofCat_f*, that is, to a* $\mathbf{U}$*-small homotopical category of cofibrant objects.*

Remark. Since $X^\flat$ has been defined as a homotopy pullback, this means that there is a factorization of $\rho_{X,lex}\colon (\mathrm{sSet}/X)_{\mathrm{lex}} \to \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ into a weak equivalence $\sigma_X\colon (\mathrm{sSet}/X)_{\mathrm{lex}} \to M$ followed by a fibration $\iota_X\colon M \to \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})$ such that, in the resulting commutative diagram

$$\begin{tikzcd}
X^\flat \arrow[r] \arrow[d, "\sigma'_X"'] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[d, "\sigma_X"] \\
A \arrow[r] \arrow[d, "\iota'_X"'] & M \arrow[d, "\iota_X"] \\
X \arrow[r, "\ell_X \text{よ}_X"] & \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

in which both inner squares are supposed to be pullback squares, σ'_X is a weak equivalence. As $\rho_{X,lex}$ is a weak equivalence, so is ι_X. By the pullback axiom, ι'_X is a trivial fibration. As $\ell_X \text{よ}_X$ is fully faithful by the preceding lemma, one can similarly show that A and $X^\flat$ are equivalent to full subcategories of M and $(\mathrm{sSet}/X)_{\mathrm{lex}}$, respectively. Hence, for the purpose of this proposition, we may assume that $X^\flat$ is a full subcategory of $(\mathrm{sSet}/X)_{\mathrm{lex}}$ and that $\varrho_X\colon X^\flat \to X$ is a trivial fibration such that

$$\begin{tikzcd}
X^\flat \arrow[r] \arrow[d, "\varrho_X"'] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[d, "\rho_{X,\mathrm{lex}}"] \\
X \arrow[r, "\ell_X \text{よ}_X"] & \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

is an actual pullback. As $X^\flat$ and $(\mathrm{sSet}/X)_{\mathrm{lex}}$ are categories, the induced commutative square

$$\begin{tikzcd}
X^\flat \arrow[r] \arrow[d, "\mathrm{ho}\,\varrho_X"'] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[d, "\mathrm{ho}\,\rho_{X,\mathrm{lex}}"] \\
\mathrm{ho}\,X \arrow[r, "\mathrm{ho}(\ell_X \text{よ}_X)"] & \mathrm{ho}\,\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

is a pullback square as well. On one hand, the homotopy category of a localization is the 1-categorical localization. Hence, $\mathrm{ho}\,\rho_{X,\mathrm{lex}}$ is a 1-categorical localization. One the other hand, there is a canonical construction for the 1-categorical localization of a model category:

Given a model category M, then its *homotopy category*[1] $\mathrm{Ho}\,M$ is the category of bifibrant objects and homotopy classes of morphisms. Additionally, the functor $M \to \mathrm{Ho}\,M$ sending each object in M to a bifibrant replacement is a 1-categorical localization.

Therefore, the 1-categorical localization $(\mathrm{sSet}/X)_{\mathrm{lex}} \to \mathrm{Ho}\,((\mathrm{sSet}/X)_{\mathrm{lex}})$, which will be denoted by Γ_X, factorizes into $\rho_{X,\mathrm{lex}}$ followed by an equivalence

[1]The reason that the notions of homotopy categories for ∞-categories and for model categories carry the same name is due to the fact that model categories present ∞-categories by localization, so that the homotopy category of the model category is (equivalent to) the homotopy category of the ∞-category it presents.

of categories $\epsilon_X \colon \mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Ho}\,((\mathrm{sSet}/X)_{\mathrm{lex}})$. In particular, $\mathrm{ho}\,X$ is equivalent to a full subcategory B of $\mathrm{Ho}\,((\mathrm{sSet}/X)_{\mathrm{lex}})$, so that there is a commutative diagram of categories

$$\begin{tikzcd}
X^\flat \arrow[r, hook] \arrow[d, "\mathrm{ho}\,\varrho_X"'] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[d, "\mathrm{ho}\,\rho_{X,\mathrm{lex}}"] \\
\mathrm{ho}\,X \arrow[r, hook] \arrow[d] & \mathrm{ho}\,\mathrm{Lex}\,(X^{\mathrm{op}}, \mathcal{S}) \arrow[d, "\epsilon_X"] \\
B \arrow[r, hook] & \mathrm{Ho}\,((\mathrm{sSet}/X)_{\mathrm{lex}})
\end{tikzcd}$$

in which the inner squares and the outer rectangle are pullback squares. Let $f \colon x \to y$ be a map in X. Per definition, $\epsilon_X(f)$ is the homotopy class of some morphism $f' \colon x' \to y'$ in $(\mathrm{sSet}/X)_{\mathrm{lex}}$, i.e. $\epsilon_X(f) = \Gamma_X(f')$. By the universal property, this means that f' belongs to $X^\flat$, that is f' is a lift of f. In particular, x' and y' are bifibrant replacements of x and y, respectively.

Proof of proposition. By the axiom of choice, we choose for each map f in X a lift f' in $X^\flat$ as above. Let S be the class of these choices. Since X is $\mathbf{U}$-small, so is S as it is bijective to the $\mathbf{U}$-small set of 1-simplices in X. Let T be the class of all objects x in $X^\flat$ for which there is a map in S of the form $x \to y$ or $y \to x$. Then there is an injection $T \to S$ sending each x in T to (a choice of) a map of the form $x \to y$ or $y \to x$. Therefore, T is $\mathbf{U}$-small. Let $C^{(0)}$ be the full subcategory of $X^\flat$ spanned by objects belonging to T. As sSet/X is locally $\mathbf{U}$-small, any of its full subcategories spanned by a $\mathbf{U}$-small set is $\mathbf{U}$-small, such as $C^{(0)}$.

Now, given a $\mathbf{U}$-small full subcategory $C^{(n)}$ of $X^\flat$ for some natural number n, let $C^{(n+1)}$ be defined to be the full subcategory of $X^\flat$ spanned by the $\mathbf{U}$-small set M of objects satisfying the following conditions:

1. All objects in $C^{(n)}$ are elements of M.
2. Given a map $f \colon a \to b$ in $C^{(n)}$ such that a is cofibrant in $X^\flat$, there is exactly one element b' in M, which is an object in $X^\flat$, such that f factors into a cofibration $a \to b'$ followed by a weak equivalence $b' \to b$ in $X^\flat$, if there is not already such an object in $C^{(n)}$.
3. Given a cospan of maps $i \colon a \to b$ and $f \colon a \to a'$ in $C^{(n)}$ such that i is a cofibration and a and a' are cofibrant in $X^\flat$, there is exactly one element $a' \sqcup_a b$ in M, which together with the appropriate inclusion maps forms a pushout of that span in $X^\flat$, if there is not already such an object in $C^{(n)}$.

Since $C^{(n)}$ is assumed to be $\mathbf{U}$-small, its hom-sets are $\mathbf{U}$-small, so that M and thus $C^{(n+1)}$ are $\mathbf{U}$-small. In particular, $C^{(n)}$ is a full subcategory of $C^{(n+1)}$. By induction, this construction gives rise to a tower of full subcategories

$$C^{(0)} \subseteq C^{(1)} \subseteq \ldots \subseteq C^{(n)} \subseteq C^{(n+1)} \subseteq \ldots \subseteq X^\flat$$

and we define C to be colimit $\bigcup_{n \in \mathbb{N}} C^{(n)}$. The induced functor $C \to X^\flat$ is fully faithful, so that we may assume that C is a full subcategory of $X^\flat$. Moreover, maps in C are declared to be weak equivalences or cofibrations if they are so in $X^\flat$. Per construction, this turns C into a homotopical category of cofibrant objects such that the inclusion $C \hookrightarrow X^\flat$ is right exact, and it remains to show that this inclusion functor is a weak equivalence. For this, we will verify the conditions of Lemma 2.22. The first condition follows immediately from the fact that the inclusion functor reflects weak equivalences per definition. Let $f\colon a \to x$ be a map in $X^\flat$ with a an object in C. Per construction, there is a map f' in C such that the images of f and f' in $\operatorname{ho} X$ are equal. As a is an object in C, it is bifibrant such that f' is of the form $a \to b$ with b a bifibrant replacement of x. Thus, there are weak equivalences $s\colon b \to x'$ and $t\colon x \to x'$ such that the diagram in $X^\flat$

$$\begin{tikzcd} a \arrow[r, "f'"] \arrow[d, "f"'] & b \arrow[d, "s"] \\ x \arrow[r, "t"'] & x' \end{tikzcd}$$

commutes. This shows the second condition of Lemma 2.22. Therefore, the inclusion functor $C \hookrightarrow X^\flat$ is a weak equivalence of homotopical categories of cofibrant objects where C is $\mathbf{U}$-small. □

Therefore, $\varrho_X\colon X^\flat \to X$ exhibits the finitely cocomplete ∞-category X as the localization of a $\mathbf{U}$-small homotopical category of cofibrant objects, and the following argument illustrates the functoriality of this construction.

4.8. Let $\mathcal{F}\colon X \to Y$ be a finitely cocontinuous functor between ($\mathbf{U}$-small) finitely cocomplete ∞-categories. On one side, recall[1] that the left Kan extension $\mathcal{F}_!\colon \operatorname{Fun}(X^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Fun}(Y^{\mathrm{op}}, \mathcal{S})$ of $\mathcal{F}$ exists such that $\mathcal{F}_! よ_X$ is canonically equivalent to $よ_Y\mathcal{F}$. Therefore, we will call $\mathcal{F}_!$ the *colimit extension* of $\mathcal{F}$. More importantly, the composite $\ell_Y よ_Y \mathcal{F}\colon X \to \operatorname{Lex}(Y^{\mathrm{op}}, \mathcal{S})$ is finitely cocomplete, so that there exists a cocontinuous functor $\mathcal{F}_{!,\mathrm{lex}}\colon \operatorname{Lex}(X^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Lex}(Y^{\mathrm{op}}, \mathcal{S})$ such that $\mathcal{F}_{!,\mathrm{lex}}\ell_X よ_X$ is equivalent to $\ell_Y よ_Y \mathcal{F}$.

On the other side, postcomposing with and pulling back along $\mathcal{F}$ induce functors $\mathcal{F}_!^\flat\colon \mathrm{sSet}/X \to \mathrm{sSet}/Y$ and $\mathcal{F}^{*,\flat}\colon \mathrm{sSet}/Y \to \mathrm{sSet}/X$, respectively, which form a Quillen adjunction $\mathcal{F}_!^\flat \dashv \mathcal{F}^{*,\flat}$ with respect to the contravariant model structure.[2] Hence, $\mathcal{F}_!^\flat$ is right exact and, in particular, homotopical as $(\mathrm{sSet}/X)_{\mathrm{rfib}}$ is a homotopical category of cofibrant objects. Therefore, $\mathcal{F}_!^\flat$ descends to a functor $\operatorname{Fun}(X^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Fun}(Y^{\mathrm{op}}, \mathcal{S})$, which is equivalent to $\mathcal{F}_!$, i.e. $\rho_Y \mathcal{F}_!^\flat$ is equivalent to $\mathcal{F}_!\rho_X$.[3] Since $\mathcal{F}_!$ itself descends to $\mathcal{F}_{!,\mathrm{lex}}$, its presentation $\mathcal{F}_!^\flat$ descends to a functor $\mathcal{F}_{!,\mathrm{lex}}^\flat\colon (\mathrm{sSet}/X)_{\mathrm{lex}} \to (\mathrm{sSet}/Y)_{\mathrm{lex}}$. More precisely, since there is a commutative diagram of the form

$$\begin{tikzcd} (\mathrm{sSet}/X)_{\mathrm{rfib}} \arrow[r, "\rho_X"] \arrow[d, "\mathcal{F}_!^\flat"] & \operatorname{Fun}(X^{\mathrm{op}}, \mathcal{S}) \arrow[r, "\ell_X"] \arrow[d, "\mathcal{F}_!"] & \operatorname{Lex}(X^{\mathrm{op}}, \mathcal{S}) \arrow[d, "\mathcal{F}_{!,\mathrm{lex}}"] & \operatorname{Lex}(X^{\mathrm{op}}, \mathcal{S})^{\simeq} \arrow[l, hook'] \arrow[d] \\ (\mathrm{sSet}/Y)_{\mathrm{rfib}} \arrow[r, "\rho_Y"] & \operatorname{Fun}(Y^{\mathrm{op}}, \mathcal{S}) \arrow[r, "\ell_Y"] & \operatorname{Lex}(Y^{\mathrm{op}}, \mathcal{S}) & \operatorname{Lex}(Y^{\mathrm{op}}, \mathcal{S})^{\simeq}, \arrow[l, hook'] \end{tikzcd}$$

$\mathcal{F}_!^\flat$ sends $(\ell_X\rho_X)^{-1}\left(\operatorname{Lex}(X^{\mathrm{op}}, \mathcal{S})^{\simeq}\right)$ to $(\ell_Y\rho_Y)^{-1}\left(\operatorname{Lex}(Y^{\mathrm{op}}, \mathcal{S})^{\simeq}\right)$. Hence, $\mathcal{F}_!^\flat$ descends to $\mathcal{F}_{!,\mathrm{lex}}^\flat$ such that $\mathcal{F}_{!,\mathrm{lex}}^\flat \ell_X^\flat$ is equivalent to $\ell_Y^\flat \mathcal{F}_!^\flat$. Moreover, since $\ell_Y\rho_Y$

[1] See [Cis19, Proposition 6.1.14].
[2] See [Cis19, Proposition 4.4.6].
[3] See [Cis19, Remark 6.1.24].

is equivalent to $\rho_{Y,\mathrm{lex}}\ell_Y^\flat$, the left adjoint of the composite Quillen adjunction $\ell_Y^\flat \mathcal{F}_!^\flat \dashv \mathcal{F}^{*,\flat}\iota_X^\flat$ sends $(\ell_X \rho_X)^{-1}\left(\mathrm{Lex}\left(X^{\mathrm{op}}, \mathcal{S}\right)^{\simeq}\right)$ to $(\rho_{Y,\mathrm{lex}})^{-1}\left(\mathrm{Lex}\left(Y^{\mathrm{op}}, \mathcal{S}\right)^{\simeq}\right)$, which is the subcategory of weak equivalences in $(\mathrm{sSet}/Y)_{\mathrm{lex}}$ because $\rho_{Y,\mathrm{lex}}$ is the localization functor. By [Hir03, Proposition 3.3.18], $\ell_Y^\flat \mathcal{F}_!^\flat$ descends to a left Quillen functor $(\mathrm{sSet}/X)_{\mathrm{lex}} \to (\mathrm{sSet}/Y)_{\mathrm{lex}}$, but this has to be $\mathcal{F}_{!,\mathrm{lex}}^\flat$ as shown above. Hence, $\mathcal{F}_{!,\mathrm{lex}}^\flat$ is right exact.

In total, there is a commutative diagram of homotopical ∞-categories of cofibrant objects

$$\begin{tikzcd}
X \arrow[r, "\ell_X \text{よ}_X"] \arrow[d, "\mathcal{F}"] & \mathrm{Lex}\left(X^{\mathrm{op}}, \mathcal{S}\right) \arrow[d, "\mathcal{F}_{!,\mathrm{lex}}"] & (\mathrm{sSet}/X)_{\mathrm{lex}} \arrow[l, "\rho_{X,\mathrm{lex}}"'] \arrow[d, "\mathcal{F}_{!,\mathrm{lex}}^\flat"] \\
Y \arrow[r, "\ell_Y \text{よ}_Y"'] & \mathrm{Lex}\left(Y^{\mathrm{op}}, \mathcal{S}\right) & (\mathrm{sSet}/Y)_{\mathrm{lex}} \arrow[l, "\rho_{Y,\mathrm{lex}}"]
\end{tikzcd}$$

and this induces a right exact functor $\mathcal{F}^\flat \colon X^\flat \to Y^\flat$ whose total left derived functor is $\mathcal{F}$ per construction.

Observation 4.9. We want to note that, instead of $(\mathrm{sSet}/X)_{\mathrm{rfib}}$, we may utilize any model category $M(X)$ presenting $\mathrm{Fun}\left(X^{\mathrm{op}}, \mathcal{S}\right)$ such that it descends to a presentation $M(X)_{\mathrm{lex}}$ of $\mathrm{Lex}\left(X^{\mathrm{op}}, \mathcal{S}\right)$, which forms a homotopical category of cofibrant objects and which is functorial in X in the following manner:

For any finitely cocontinuous functor $\mathcal{F} \colon X \to Y$ between finitely cocomplete ∞-categories, there is a right exact functor $\mathcal{F}_{!,\mathrm{lex}}^\flat \colon M(X)_{\mathrm{lex}} \to M(Y)_{\mathrm{lex}}$ such that

$$\begin{tikzcd}
X \arrow[r, "\ell_X \text{よ}_X"] \arrow[d, "\mathcal{F}"] & \mathrm{Lex}\left(X^{\mathrm{op}}, \mathcal{S}\right) \arrow[d, "\mathcal{F}_{!,\mathrm{lex}}"] & M(X)_{\mathrm{lex}} \arrow[l] \arrow[d, "\mathcal{F}_{!,\mathrm{lex}}^\flat"] \\
Y \arrow[r, "\ell_Y \text{よ}_Y"] & \mathrm{Lex}\left(Y^{\mathrm{op}}, \mathcal{S}\right) & M(Y)_{\mathrm{lex}} \arrow[l]
\end{tikzcd}$$

is a commutative diagram in $\mathrm{CofQCat}_f$.

Remark. In fact, using the model of simplicial presheaves as in Lemma 4.14 and the arguments of the last subsection, it is possible to show that any not necessarily functorial presentation of finitely cocomplete ∞-categories by (ordinary) categories suffices to construct a presentation by homotopipcal categories of cofibrant objects, which is functorial in the aforementioned sense.

Homotopically Right Exact Functors

Given a finitely cocomplete ∞-category X, a homotopical category D of cofibrant objects and a finitely cocontinuous functor $\mathcal{F} \colon X \to LD$, in the preceding subsection, we have constructed a commutative square of finitely cocomplete ∞-categories of the form

$$\begin{tikzcd}
X \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{E}"'] & LD \\
L\left(X^\flat\right) \arrow[r, "\mathbf{L}(\mathcal{F}^\flat)"] & L\left((LD)^\flat\right) \arrow[u, "\mathcal{H}"']
\end{tikzcd}$$

where $\mathcal{E}$ and $\mathcal{H}$ are equivalence of ∞-categories. In order to solve the rectification problem, it suffices to exhibit $\mathcal{H}$ as the image of some map $H \colon (LD)^\flat \to D$ under

$\bar{L}\colon L\mathrm{CofCat}_f \to \mathrm{RexCat}_\infty$, i.e. we have reduced the rectification problem for finitely cocontinuous functors (with arbitrary domain) to that of equivalences between localizations of objects in CofCat_f. So, we may assume without loss of generality that $\mathcal{F}$ is an equivalence of ∞-categories of the form $LC \to LD$ for some homotopical categories C and D of cofibrant objects.

Observation 4.10. Let $\mathcal{F}, \mathcal{G}\colon X \to Y$ be finitely cocontinuous functors between finitely cocomplete ∞-categories such that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic in $\mathrm{Fun}(X, Y)$. If X and Y are presented by quasi-categories, then there is a commutative triangle of quasi-categories of the form

$$\begin{tikzcd} & \mathrm{Fun}(J, Y) \arrow[dr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & \\ X \arrow[ur, "H"] \arrow[rr, "(\mathcal{F}, \mathcal{G})"'] & & Y \times Y. \end{tikzcd} \tag{4.10.1}$$

This datum is commonly known as a right homotopy from $\mathcal{F}$ to $\mathcal{G}$ with respect to the Joyal model structure on simplicial sets as the commutative triangle of quasi-categories

$$\begin{tikzcd} & \mathrm{Fun}(J, Y) \arrow[dr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & \\ Y \arrow[ur, "\Delta_J"] \arrow[rr, "(1_Y, 1_Y)"'] & & Y \times Y \end{tikzcd} \tag{4.10.2}$$

forms a path object of Y in this model structure. In particular, this implies that $\mathcal{F}$ and $\mathcal{G}$ represent the same map in $\mathrm{ho}\,\mathrm{Cat}_\infty$ by Lemma 1.11 citing [Cis19, Proposition 7.6.14]. In other words, naturally isomorphic functors of quasi-categories are homotopic.

We would like to show that this also implies that the induced maps in $\mathrm{ho}\,\mathrm{RexCat}_\infty$ are equal. First, note that the commutative triangle in (4.10.2) is a path object of Y in RexQCat as well. This has been shown in Lemma 3.7, but to repeat: Because Δ_J is a weak categorical equivalence between quasi-categories and because colimits in $\mathrm{Fun}(J, Y)$ are componentwise so, both Δ_J and $(\mathrm{ev}_0, \mathrm{ev}_1)$ preserve and reflect colimits, in particular finite ones. It is worth mentioning that the product of Y with itself in sSet is the underlying quasi-category of the product of Y with itself in RexQCat[1], so that there is a canonical structure on $Y \times Y$ such that $(1_Y, 1_Y)$ and $(\mathcal{F}, \mathcal{G})$ are right exact functors. Second, as noted, $(\mathrm{ev}_0, \mathrm{ev}_1)$ reflects colimits, so that the right exactness of $(\mathcal{F}, \mathcal{G})$ implies that of H. Thus, the commutative triangle in (4.10.1) also forms a right homotopy in RexQCat, so that $\mathcal{F}$ and $\mathcal{G}$ represent the same map in $\mathrm{ho}\,\mathrm{RexCat}_\infty$.

The preceding observation allows us to conclude the rectification problem by exhibiting equivalences $\mathcal{F}\colon LC \to LD$ between localizations of homotopical categories of cofibrant objects as (cospans of) total right derived functors from C to D, disregarding the specific presentation of total right derived functors given by Lemma 3.10. Therefore, as the next step, we consider solutions to the filling problem

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d, "\gamma_C"] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, "\mathcal{F}"] & LD \end{tikzcd}$$

[1] See the construction in Lemma 2.23.

which are not necessarily right exact. In Lemma 4.22, it will be shown that these already provide us with solutions to the rectification problem. This leads us to the notion of homotopically right exact functor and, although these can be defined as Kan lifts, dually to Kan extensions, the following is going to suffice.

Definition 4.11. [Cis10, Paragraph 4.1] A functor $F\colon C \to D$ between homotopical categories of cofibrant objects is *homotopically right exact* if the following conditions hold:

- For any weak equivalences $f\colon a \to b$ in C, the map $F(f)\colon F(a) \to F(b)$ in D is a weak equivalence.
- The canonical map $\varnothing_D \to F(\varnothing_C)$ in D is a weak equivalence.
- For any *homotopy pushout square* in C

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & a' \arrow[d, "g"] \\ b \arrow[r, "v"'] & b', \end{tikzcd}$$

  the induced commutative square in D

$$\begin{tikzcd} F(a) \arrow[r, "F(u)"] \arrow[d, "F(f)"'] & F(a') \arrow[d, "F(g)"] \\ F(b) \arrow[r, "F(v)"'] & F(b') \end{tikzcd}$$

  is a homotopy pushout square.

In other words, F is homotopically right exact if F preserves

- weak equivalences;
- initial objects up to weak equivalence;
- homotopy pushout squares.

Dually, a functor $F\colon C \to D$ between homotopical categories of fibrant objects is *homotopically left exact* if $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is homotopically right exact.

Proposition 4.12. *Let $F\colon C \to D$ be an arbitrary functor between homotopical categories of cofibrant objects. Then F is homotopically right exact if and only if there is a finitely cocontinuous functor $\mathcal{F}\colon LC \to LD$ such that the diagram of ∞-categories*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, "\mathcal{F}"'] & LD \end{tikzcd} \tag{4.12.1}$$

commutes. In particular, $\mathcal{F}$ is a total (left) derived functor of F and F is (automatically) homotopical.

Proof. Let F be homotopically right exact. Then $\gamma_D F$ sends weak equvialences to isomorphisms, so that the functor $\mathcal{F}\colon LC \to LD$ induced by the universal property of the localization γ_C is a total left derived functor of F by Lemma 2.3. Hence, there is a commutative diagram of ∞-categories of the form (4.12.1).

Conversely, let $\mathcal{F}\colon LC \to LD$ be a finitely cocontinuous functor such that the diagram in (4.12.1) commutes. Initial objects, homotopy pushout squares and weak equivalences in C and D are by definition those objects, diagrams and maps that are sent to initial objects, pushout squares and isomorphisms in LC and LD, respectively. Thus, due to commutativity of the diagram in (4.12.1), the finite cocontinuity of $\mathcal{F}$ implies the homotopical right exactness of F.

Moreover, as $\gamma_C^*\colon \mathrm{Fun}(LC, LD) \to \mathrm{Fun}(C, D)$ is fully faithful, we get the following computation:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(LC,LD)}(-,\mathcal{F}) &\simeq \mathrm{Hom}_{\mathrm{Fun}(C,LD)}\left(\gamma_C^*(-), \gamma_C^*(\mathcal{F})\right)\\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(C,LD)}\left(\gamma_C^*(-), \gamma_D F\right).\end{aligned}$$

By the Yoneda lemma, $\mathcal{F}$ is a total left derived functor of F. □

Example 4.13. An immediate example of homotopically right exact functors are right exact functors between homotopical categories of cofibrant objects since, by Lemma 1.25, all homotopy pushout square are computed by pushouts of cofibrations.

The preceding proposition states that homotopically right exact functors are precisely those homotopical functors that present finitely cocontinuous functors. In order to show that homotopical right exact solutions to the rectification problem for equivalences provide (strictly) right exact solutions, we will make use of another presentation of the space-valued presheaf category, namely the simplicial presheaf category. Our main reference for the subsequent paragraphs is [Cis10].

Definition 4.14. [Cis10, Paragraph 3.1] Let C be a homotopical category of fibrant objects. Let $P(C)$ denote the projective model structure on the category $\mathrm{Fun}\left(C^{\mathrm{op}}, \mathrm{sSet}_{\mathrm{KQ}}\right)$, i.e. the weak equivalences and fibrations are defined to be componentwise weak homotopy equivalences and Kan fibrations, respectively. Let $\text{ヨ}_C\colon C \to P(C)$ denote the 1-categorical Yoneda embedding where sets are regarded as constant simplicial sets.[1] Lastly, let

$$\begin{tikzcd} P(C) \arrow[r, shift left, "\gamma^\flat_{C,!}"] \arrow[r, phantom, "\perp"] & P_{wC}(C) \arrow[l, shift left, "\gamma_C^{*,\flat}"] \end{tikzcd}$$

denote the left Bousfield localization of $P(C)$ at the (essential) image of the subcategory wC of weak equivalences in C under ヨ_C. In particular, $P_{wC}(C)$ is equivalent to the full subcategory of $P(C)$ spanned by simplicial presheaves that send weak equivalences in C to weak homotopy equivalences.

[1] ヨ is the first letter of the name "Yoneda" in Katakana, the more angular of the two Japanese syllabaries. This seems fitting as the notion of sets can be thought of as more rigid than that of spaces. Hence, the space-valued Yoneda embedding is denoted by the rounder Hiragana letter よ.

Remark. By [Cis19, Proposition 7.11.13], $P_{wC}(C)$ is equivalent to a nerve of a category, so that it may be regarded as the model category theoretical left Bousfield localization of $P(C)$, and, by [Cis10, Théorème 3.2], it is a (right) proper model category and the class of weak equivalences in $P_{wC}(C)$ is closed under finite products. This allows us to equip $P_{wC}(C)$ with the structure of a homotopical category of fibrant objects.

Lemma 4.15. *Let M be a right proper model category. Then its underlying category together with the class wM of weak equivalences and the class of wM-*fibrations[1] *forms a category with weak equivalences and fibrations. If wM is addtionally closed under finite products, we have a homotopical category of fibrant objects.*

Proof. Because model categories are (co-)complete, M admits a terminal object, and, because the class of weak equivalences remains unchanged, it forms a subcategory and satisfies the 2-out-of-3 property. In fact, wM satisfies the 2-out-of-6 property due to saturation. As commutative squares of the form

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, "1_{x'}"'] & x \arrow[d, "1_x"] \\ x' \arrow[r, "f"'] & x \end{tikzcd}$$

are pullback squares, all identity morphisms in M are wM-fibrations. Since homotopy pullback squares are those diagrams that are pullback squares in the localization, there is a pasting law for homotopy pullbacks. With this, one can show that wM-fibrations are closed under compositions and satisfy the pullback axiom. As weak equivalences in M become isomorphisms in LM, and, as the class of isomorphisms is closed under pullbacks, the pullback axiom for trivial fibrations follows immediately from the above. Lastly, it remains to show the factorization axiom. As any morphism in M factors into a weak equivalence followed by a fibration, it suffices to show that any fibration of the model structure is already a wM-fibration. Since the class of fibrations is closed under pullbacks, it suffices to show that the class of weak equivalences is closed under pullbacks along fibrations, but this holds for M precisely because it is right proper.

Per definition, the wM-fibrant objects are those objects x in M such that, for any weak equivalence $v\colon y \to z$, the map $x \times y \to x \times z$ induced by 1_x and v is a weak equivalence. Hence, if the class of weak equivalences is closed under finite products, any object is wM-fibrant. □

Proposition 4.16. *[Cis10, Remarque 3.13] Let C be a homotopical category of fibrant objects and equip $P_{wC}(C)$ with the structure of a homotopical category of fibrant objects as above. Then the composite functor $\gamma^\flat_{C,!}\exists_C\colon C \to P_{wC}(C)$ is left exact.*

For the proof, we need the following.

Lemma 4.17. *Let C be an ∞-category with two subcategories V and W and let $\gamma_W\colon C \to L_W C$ and $\gamma_V\colon C \to L_V$ denote the respective localization functors.*

[1] See Lemma 1.17.

Then the total derived functor of γ_W where C is equipped with V and $L_W C$ with $\gamma_W(V)$ is a localization functor of $L_V C$ at $\gamma_V(W)$.

Remark. Note that this statement is symmetric in V and W. Informally, this means that the order of subcategories at which we localize C does not matter.

Proof. Let $\gamma_{W,V}\colon L_W C \to L_{W,V} C$ denote the localization functor of $L_W C$ at $\gamma_W(V)$ and let $\overline{\gamma_W}\colon L_V C \to L_{W,V} C$ be the total derived functor of γ_W with respect to γ_V and $\gamma_{W,V}$. Then the square of ∞-categories

$$\begin{tikzcd} C \arrow[r, "\gamma_W"] \arrow[d, "\gamma_V"'] & L_W C \arrow[d, "\gamma_{W,V}"] \\ L_V C \arrow[r, "\overline{\gamma_W}"'] & L_{W,V} C \end{tikzcd}$$

commutes. If presented by quasi-categories, such a commutative square exists on the level of simplicial sets by the construction of Lemma 3.10 since γ_W clearly sends V to $\gamma_W(V)$. From the commutativity of this square, one can immediately derived that $\overline{\gamma_W}$ is essentially surjective since γ_V, γ_W and $\gamma_{W,V}$ are assumed to be so, and that $\overline{\gamma_W}$ sends $\gamma_V(W)$ to isomorphisms. Moreover, for any ∞-category D, there is an induced commutative square of ∞-categories of the form

$$\begin{tikzcd} \operatorname{Fun}(L_{W,V} C, D) \arrow[r, "\overline{\gamma_W}^*"] \arrow[d, "(\gamma_{W,V})^*"'] & \operatorname{Fun}(L_V C, D) \arrow[d, "(\gamma_V)^*"] \\ \operatorname{Fun}(L_W C, D) \arrow[r, "(\gamma_W)^*"'] & \operatorname{Fun}(C, D), \end{tikzcd}$$

so that $\overline{\gamma_W}^*$ is fully faithful since $(\gamma_V)^*$, $(\gamma_W)^*$ and $(\gamma_{W,V})^*$ are assumed to be so. Lastly, it remains to show that the essential image of $\overline{\gamma_W}^*$ is spanned by those functors $F\colon L_V C \to D$ that send $\gamma_V(W)$ to isomorphisms. For this, let F be such a functor. Then $(\gamma_V)^*(F)$ sends W to isomorphisms, so that it is naturally isomorphic to a functor of the form $(\gamma_W)^*(G)$ where G is a functor from $L_W C$ to D. Clearly, $(\gamma_V)^*(F)$ sends V to isomorphisms and, thus, so does $(\gamma_W)^*(G)$. Therefore, G is naturally isomorphic to a functor of the form $(\gamma_{W,V})^*(H)$ where H is a functor from $L_{W,V} C$ to D. Hence, $(\gamma_V)^*(F)$ is naturally isomorphic to $(\gamma_W)^*(\gamma_{W,V})^*(H)$, which equals $(\gamma_V)^*(\gamma_W)^*(H)$ due to the commutativity of the preceding square. Because $(\gamma_W)^*$ is fully faithful, this implies that F is naturally isomorphic to $(\gamma_W)^*(H)$, which concludes the proof. ☐

Observation 4.18. It is commonly known that the global model structures on simplicial presheaves, such as the projective one, present the space-valued presheaf ∞-category. A proof of this is given in [Cis19, Theorem 7.9.8] stating that such a localization functor $\rho_C\colon P(C) \to \operatorname{Fun}(C^{\mathrm{op}}, \mathcal{S})$ is given by postcomposing with the localization functor $\mathrm{sSet}_{KQ} \to \mathcal{S}$. By the preceding lemma, the localization of $P_{wC}(C)$ is (equivalent to) the localization of $\operatorname{Fun}(C^{\mathrm{op}}, \mathcal{S})$ at the (essential) image of wC under $\rho_C \exists_C$. Since C is a category, the set-valued Yoneda embedding already models the space-valued one, so that $\ell_C \exists_C$ is equivalent to よ_C. Since localizations of $\operatorname{Fun}(C^{\mathrm{op}}, \mathcal{S})$ are left Bousfield localizations,[1] the localization of P_{wC} is (equivalent to) the full subcategory of $\operatorname{Fun}(C^{\mathrm{op}}, \mathcal{S})$ spanned by the so-called $\text{よ}_C(wC)$*-local objects*[2]. Using the Yoneda lemma, one

[1] See [Cis19, Proposition 7.11.4].
[2] See [Cis19, Paragraph 7.11.3]

can check that these $\text{よ}_C(wC)$-local objects are precisely those presheaves that send weak equivalences to isomorphisms. Therefore, $\mathrm{Fun}_{wC^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S})$ is the aforementioned subcategory. By the preceding lemma, $\gamma^{\flat}_{C,!} \colon P(C) \to P_{wC}(C)$ descends to a localization functor of the form $\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}_{wC^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S})$. Since the total derived functors of adjoint homotopical functors are again adjoint,[1] the right Quillen functor $\gamma^{*,\flat}_{C} \colon P_{wC}(C) \to P(C)$ descends to the inclusion $\mathrm{Fun}_{wC^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S}) \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$. Recall that $\mathrm{Fun}_{wC^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S})$ is equivalent to $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ via γ^*_C, so that, as the notation suggests, $\gamma_{C,!}$ and γ^*_C are total derived functor of $\gamma^*_{C,!}$ and $\gamma^{*,\flat}_C$, respectively.

Remark. Let $\rho_{C,wC} \colon P_{wC}(C) \to \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ denote the localization functor. Then the diagrams of ∞-categories

$$\begin{tikzcd} P(C) \arrow[r, "\gamma^{\flat}_{C,!}"] \arrow[d, "\rho_C"'] & P_{wC}(C) \arrow[d, "\rho_{C,wC}"] \\ \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}) \arrow[r, "\gamma_{C,!}"'] & \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P(C) \arrow[d, "\rho_C"'] & P_{wC}(C) \arrow[l, "\gamma^{*,\flat}_{C}"'] \arrow[d, "\rho_{C,wC}"] \\ \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}) & \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S}) \arrow[l, "\gamma^*_C"] \end{tikzcd}$$

commute. Moreover, due to the computation in Cat_∞

$$\rho_{C,wC}\gamma^{\flat}_{C,!}\exists_c \simeq \gamma_{C,!}\rho_C\exists_C \simeq \gamma_{C,!}\text{よ}_C \simeq \text{よ}_{LC}\gamma_C \tag{4.18.1}$$

the diagram of ∞-categories

$$\begin{tikzcd} C \arrow[r, "\gamma^{\flat}_{C,!}\exists_C"] \arrow[d, "\gamma_C"'] & P_{wC}(C) \arrow[d, "\rho_{C,wC}"] \\ LC \arrow[r, "\text{よ}_{LC}"'] & \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S}) \end{tikzcd}$$

commutes, i.e. よ_{LC} is a total derived functor of $\gamma^{\flat}_{C,!}\exists_C$. For this reason, we simplify the notation $\gamma^{\flat}_{C,!}\exists_C$ to $\exists_{LC}$ as we want to see it as a presentation of よ_{LC}. By Lemma 4.12, $\exists_{LC}$ is homotopically right exact and, in particular, preserves homotopy pullbacks, e.g. pullbacks of fibrations by Lemma 1.25.

Proof of Lemma 4.16. [Cis10, Corollaire 3.12] Per construction, $\exists_{LC}$ preserves (finite) colimits and weak equivalences, so that it remains to show preservation of fibrations. For this, let $p \colon a \to b$ be a fibration in C and denote $\exists_{LC}(p)$ by $f \colon x \to y$. Then we would like to show that any pullback square in $P_{wC}(C)$ of the form

$$\begin{tikzcd} F \times_y x \arrow[r] \arrow[d] & x \arrow[d, "f"] \\ F \arrow[r, "\epsilon"'] & y \end{tikzcd} \tag{4.18.2}$$

is a homotopy pullback square. By [Cis06, Proposition 3.4.48], we can equivalently show that, for any object z in $P_{wC}(C)$ with $z = \exists_{LC}(c)$ for some object

[1] See [Cis19, Proposition 7.1.14] or [Cis19, Theorem 7.5.30].

c in C and any map $s\colon z \to F$, the induced pullback square in $P_{wC}(C)$

$$\begin{tikzcd} z \times_y x \arrow[r] \arrow[d] & x \arrow[d, "f"] \\ z \arrow[r, "\epsilon s"] & y \end{tikzcd} \tag{4.18.3}$$

is a homotopy pullback. Since $\exists_{LC}$ is fully faithful[1], this diagram comes from a pullback square in C of the form

$$\begin{tikzcd} c \times_b a \arrow[r] \arrow[d] & a \arrow[d, "p"] \\ c \arrow[r] & b \end{tikzcd}$$

which is a homotopy pullback square because p is a fibration. By the preceding observation, $\exists_{LC}$ preserves homotopy pullbacks, so that the diagram in (4.18.3) is a homotopy pullback square. This shows that f is a wC-fibration. $\square$

Remark. The proof idea of [Cis06, Proposition 3.4.48] goes as follows: The pullback square in (4.18.2) is a homotopy pullback square if and only if, for any factorization of f into a weak equivalence S followed by a fibration P and any induced diagram in $P_{wC}(C)$

$$\begin{tikzcd} F \times_y x \arrow[r] \arrow[d, "S'"] & x \arrow[d, "S"] \\ F \times_y G \arrow[r] \arrow[d] & G \arrow[d, "P"] \\ F \arrow[r] & y \end{tikzcd}$$

where the inner squares are pullback squares, the map S' is a weak equivalence. Given a map $s\colon z \to F$ in $P_{wC}(C)$ as in the proof above, the map S'_z in the induced diagram

$$\begin{tikzcd} z \times_y x \arrow[r] \arrow[d, "S'_z"] & F \times_y x \arrow[r] \arrow[d, "S'"] & x \arrow[d, "S"] \\ z \times_y G \arrow[r] \arrow[d] & F \times_y G \arrow[r] \arrow[d] & G \arrow[d, "P"] \\ z \arrow[r, "s"] & F \arrow[r] & y \end{tikzcd}$$

where each inner square is a pullback square is a weak equivalence if and only if the square in (4.18.3) is a homotopy pullback. Hence, our argument boils down to the fact that the map S' over F is a weak equivalence if and only if it is "fiberwise" so. This is rigorously proven in [Cis06, Corollaire 3.4.47] and we would like to give an informal picture:

[1] $\exists_C$ is clearly fully faithful, and, as part of a Bousfield localization of model categories, $\gamma^\flat_{C,!}$ is presented by the identity of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{sSet})$.

By the so-called co-Yoneda lemma, any presheaf F in $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ is a colimit of representables:

$$F \simeq \operatorname*{colim}_{s \in F(c)} z.$$

Moreover, these colimits are stable under pullback[1], so that the map

$$S' \colon F \times_y x \cong \operatorname*{colim}_{s \in F(c)} (z \times_y x) \longrightarrow F \times_y G \cong \operatorname*{colim}_{s \in F(c)} (z \times_y G)$$

is given precisely by $S'_z \colon z \times_y x \to z \times_y G$ for each $s \in F(c)$. Therefore, if each S'_z is an isomorphism in $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$, i.e. a weak equivalence in $P_{wC}(C)$, so is S'.

Definition 4.19. Let C be a homotopical category of fibrant objects. An object F in $P_{wC}(C)$ is called *weakly representable* if the presheaf $\rho_{C,wC}(F) \colon LC^{\mathrm{op}} \to \mathcal{S}$ is representable. Let $P_{wC}(C)_{\mathrm{wrep}}$ be the full subcategory of $P_{wC}(C)$ spanned by the weakly representable presheaves and let $P_{wC}(C)_{\mathrm{wrf}}$ be the full subcategory spanned by weakly representable presheaves that are fibrant in $P(C)$, i.e. componentwise Kan complexes.

Observation 4.20. By $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$, we denote the full subcategory of $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ spanned by the representable presheaves. Per definition, there is a pullback square of ∞-categories of the form

$$\begin{tikzcd} P_{wC}(C)_{\mathrm{wrep}} \arrow[r, hook] \arrow[d] & P_{wC}(C) \arrow[d, "\rho_{C,wC}"] \\ \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}} \arrow[r, hook] & \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S}). \end{tikzcd}$$

Since the inclusion $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}} \hookrightarrow \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ is an isofibration, this diagram can be presented by a pullback square of simplicial sets. More importantly, as the Yoneda embedding is continuous, $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$ is finitely complete and canonically carries a structure of homotopical quasi-categories of fibrant objects such that its inclusion functor into $\mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ is left exact. Recall that $\rho_{C,wC}$ is left exact as a localization functor of a homotopical category of fibrant objects, so that, by the dual version of Lemma 2.23, $P_{wC}(C)_{\mathrm{rep}}$ can be equipped with a structure of a homotopical (quasi-)category of fibrant objects such that the above diagram is a pullback square in FibQCat. By the pullback axiom (in $\mathrm{FibQCat}_f$), the one projection $P_{wC}(C)_{\mathrm{wrep}} \to P_{wC}(C)$, which is (equivalent to) an inclusion, is a fibration in FibQCat, so that, by the dual version of Lemma 1.16, the other projection $P_{wC}(C)_{\mathrm{wrep}} \to \mathrm{Fun}(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$ is a weak equivalence in FibQCat, thus a localization functor. In particular, this is a homotopy pullback square.

Analogously, there is a commutative diagram (of subcategories of $P(C)$) in FibQCat of the form

$$\begin{tikzcd} P_{wC}(C)_{\mathrm{wrf}} \arrow[r, hook] \arrow[d, hook] & P_{wC}(C) \cap P(C)_f \arrow[r, hook] \arrow[d, hook] & P(C)_f \arrow[d, hook] \\ P_{wC}(C)_{\mathrm{wrep}} \arrow[r, hook] & P_{wC}(C) \arrow[r, hook, "\gamma_C^{*,\flat}"] & P(C) \end{tikzcd}$$

[1] Recall that colimits in ∞-topoi are universal, that $\mathcal{S}$ is an ∞-topoi, and that limits and colimits in presheaf ∞-categories are componentwise.

where each inner square is a pullback square. Because $P_{wC}(C)_{\mathrm{wrep}} \hookrightarrow P_{wC}(C)$ is a fibration, so is $P_{wC}(C)_{\mathrm{wrf}} \hookrightarrow P_{wC}(C) \cap P(C)_f$ and, in particular, the left inner square is a homotopy pullback square. One can quickly show that $\gamma_C^{*,\flat}$ is a fibration in FibCat. Hence, $P_{wC}(C) \cap P(C)_f \hookrightarrow P(C)_f$ is a fibration and the right inner square is a homotopy pullback square as well. By Lemma 1.16, the canonical functors $P_{wC}(C) \cap P(C)_f \hookrightarrow P_{wC}(C)$ and $P_{wC}(C)_{\mathrm{wrf}} \hookrightarrow P_{wC}(C)_{\mathrm{wrep}}$ are weak equivalences in FibCat.

Remark. Up to homotopy, the projection $P_{wC}(C)_{\mathrm{wrep}} \to \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$ is the restriction of $\rho_{C,wC} \colon P_{wC}(C) \to \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})$. Therefore, by abuse of notation, we would like to denote it by $\rho_{C,wC}$ as well. We also want to point out again that $\rho_{C,wC} \colon P_{wC}(C)_{\mathrm{wrep}} \to \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$ is a localization functor as argued above.

Definition 4.21. By the formula in (4.18.1), the diagram of ∞-categories

$$\begin{tikzcd} C \arrow[r, "\exists_{LC}"] \arrow[d, "よ_C \gamma_C"'] & P_{wC}(C) \arrow[d, "\rho_{C,wC}"] \\ \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}} \arrow[r, hook] & \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S}) \end{tikzcd}$$

commutes, so that it corresponds to a functor $C \to P_{wC}(C)_{\mathrm{wrep}}$ such that

$$\begin{tikzcd} & P_{wC}(C)_{\mathrm{wrep}} \arrow[dr, "\rho_{C,wC}"] & \\ C \arrow[ur] \arrow[rr, "よ_{LC}\gamma_C"'] & & \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & P_{wC}(C)_{\mathrm{wrep}} \arrow[dr, hook] & \\ C \arrow[ur] \arrow[rr, "\exists_{LC}"'] & & P_{wC}(C) \end{tikzcd}$$

are commutative triangles of ∞-categories.

By the commutativity of the right triangle, $C \to P_{wC}(C)_{\mathrm{wrep}}$ is the restriction of $\exists_{LC} \colon C \to P_{wC}(C)$ up to homotopy. Therefore, we will denote it by $\exists_{LC}$ as well. Moreover, the commutativity of the left triangle implies that the equivalence of ∞-categories $よ_{LC} \colon LC \to \mathrm{Fun}\,(LC^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$ is a total derived functor of $\exists_{LC} \colon C \to P_{wC}(C)_{\mathrm{wrep}}$. Hence, the latter is a weak equivalence in FibCat.

Remark. Although the terms "∞-category" and "quasi-category" are interchangable for some of our sources, such as [Cis19] and [Lur18], we have been and will be using the former for a conceptual and schematic notion and the latter for a concrete model or presentation of it. Hence, "∞-category" refers to objects of homotopy theories, such as Cat_∞, RexCat_∞ etc., whereas "quasi-category" refers to objects of (model) categories, such as $\mathrm{sSet}_{\mathrm{Joyal}}$, RexQCat and so on.

In the case of the previous definition, this means that, if presented by quasi-categories, the commutative diagrams of ∞-categories are modelled by functors of quasi-categories, whose composites are naturally isomorphic instead of (strictly) equal.

A Rectification of Equivalences

Let C and D be homotopical categories of cofibrant objects. Assuming that, for each equivalence of ∞-categories of the form $\mathcal{F} \colon LC \to LD$, there is a commutative square of ∞-categories of the form

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & D \arrow[d, "\gamma_D"] \\ LC \arrow[r, "\mathcal{F}"'] & LD, \end{tikzcd}$$

then F is homotopically right exact and $\mathcal{F}$ is its total (left) derived functor by Lemma 4.12. Under this assumption, the following paragraph demonstrates how to replace F by a zig-zag of (strictly) right exact functors which $\bar{L}\colon \mathrm{LCofQCat}_f \to \mathrm{RexCat}_\infty$ sends to $\mathcal{F}$.

4.22. In actuality, we show the dual statement. So let $F\colon C \to D$ be a homotopically left exact functors between homotopical categories of fibrant objects such that $\mathbf{R}F\colon LC \to LD$ is an equivalence of ∞-categories. Then precomposition with F induces an adjunction

$$\begin{tikzcd} P(C) \arrow[r, shift left, "F_!"] \arrow[r, phantom, "\perp"] & P(D) \arrow[l, shift left, "F^*"] \end{tikzcd}$$

which is a Quillen adjunction as F^* preserves componentwise fibrations and trivial fibrations, and we would like to show that this adjunction descends to $P_{wC}(C)$ and $P_{wD}(D)$. Since the underlying functors of left Bousfield localizations of model categories are identities, the composites

$$\begin{tikzcd} P_{wC}(C) \arrow[r, shift left, "\gamma^\flat_{D,!}F_!\gamma^{*,\flat}_C"] \arrow[r, phantom, "\perp"] & P_{wD}(D) \arrow[l, shift left, "\gamma^\flat_{C,!}F^*\gamma^{*,\flat}_D"] \end{tikzcd}$$

form an adjunction, and we are left to show that the left adjoint preserves trivial cofibrations and the right adjoint preserves trivial fibrations. Because the class of cofibrations and, thus, the class of trivial fibrations of a model category and its left Bousfield localization are the same, the left adjoint $\gamma^\flat_{D,!}F_!\gamma^{*,\flat}_C$ preserves cofibrations and the right adjoint $\gamma^\flat_{C,!}F^*\gamma^{*,\flat}_D$ preserves trivial fibrations. Thus, it suffices to show that the left adjoint preserves weak equivalences. To do so, we show that $F_!$ sends $\exists_C(p)\colon \exists_C(x) \to \exists_C(y)$ for any weak equivalence $p\colon x \to y$ in C to a map of the form $\exists_D(q)\colon \exists_D(a) \to \exists_D(b)$ for some weak equivalence $q\colon a \to b$ in D. Since F preserves weak equivalences, it suffices to show that $F_!\exists_C$ is isomorphic to $\exists_D F$. If we denote the Set-valued Yoneda embedding of $C \times \Delta$ by

$$h_{C\times\Delta}\colon C \times \Delta \longrightarrow \mathrm{Fun}\left((C\times\Delta)^{\mathrm{op}}, \mathrm{Set}\right)$$

and if we denote by i_C the functor $C \to C\times\Delta$ given by the identity of C and the constant functor $C \to \Delta$ with value $[0]$, then one can check that $\exists_C \cong h_{C\times\Delta} i_C$ by using the fact that $\mathrm{Fun}\left((C\times\Delta)^{\mathrm{op}}, \mathrm{Set}\right) \cong \mathrm{Fun}\left(C^{\mathrm{op}}, \mathrm{sSet}\right)$. Under this identification, we also get $F^* = (F\times \mathrm{id}_\Delta)^*$ as functors from $\mathrm{Fun}\left((D\times\Delta)^{\mathrm{op}}, \mathrm{Set}\right)$ to $\mathrm{Fun}\left((C\times\Delta)^{\mathrm{op}}, \mathrm{Set}\right)$, so that $F_! h_{C\times\Delta} \cong h_{D\times\Delta}(F\times\mathrm{id}_\Delta)$. By precomposition with i_C, we get $F_!\exists_C \cong h_{D\times\Delta} i_D F \cong \exists_D F$ because of $i_D F = (F\times\mathrm{id}_\Delta) i_C$. Hence, we get $F_!\exists_C(p) \cong \exists_D F(p)$.

For the sake of simplicity, we write $F_!$ for $\gamma^\flat_{D,!}F_!\gamma^{*,\flat}_C$ and F^* for $\gamma^\flat_{C,!}F^*\gamma^{*,\flat}_D$. Since $F^*\colon P_{wD}(D) \to P_{wC}(C)$ is a right Quillen functor, it preserves weak equivalences (and fibrations) between fibrant objects. Hence, F^* forms a left exact functor with respect to the structures of homotopical categories of fibrant objects as in Lemma 4.15 if restricted to a functor of the form

$$F^* \colon P_{wD}(D) \cap P(D)_f \longrightarrow P_{wC}(C) \cap P(C)_f$$

and we would like to show that F^* restricts further to $P_{wD}(D)_{\mathrm{wrf}} \to P_{wC}(C)_{\mathrm{wrf}}$ by observing that it preserves weakly representable presheaves. For this, we exhibit $\mathbf{R}F^* \colon \operatorname{Fun}(LD^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Fun}(LC^{\mathrm{op}}, \mathcal{S})$ as the colimit extension of some functor $\mathcal{G} \colon LD \to LC$ as then the diagram of ∞-categories

$$\begin{tikzcd}
LC \arrow[r, hook, "\text{よ}_{LC}"] & \operatorname{Fun}(LC^{\mathrm{op}}, \mathcal{S}) & P_{wC}(C) \arrow[l, hook', "\rho_{C,wC}"'] \\
LD \arrow[u, "\mathcal{G}"] \arrow[r, hook, "\text{よ}_{LD}"] & \operatorname{Fun}(LD^{\mathrm{op}}, \mathcal{S}) \arrow[u, "\mathbf{R}F^*"] & P_{wD}(D) \arrow[l, hook', "\rho_{D,wD}"'] \arrow[u, "F^*"]
\end{tikzcd}$$

commutes, which induces a functor $P_{wD}(D)_{\mathrm{wrep}} \to P_{wC}(C)_{\mathrm{wrep}}$ that is the restriction of F^* (up to homotopy) by Lemma 4.20. We would like to argue why any homotopy inverse to $\mathbf{R}F$ is an appropriate candidate for such a $\mathcal{G}$. Therefore, let $\mathcal{G}$ be a homotopy inverse to $\mathbf{R}F$. Then it can be promoted to a right adjoint of $\mathbf{R}F$,[1] so that $\mathcal{G}_!$ and $(\mathbf{R}F)^*$ are equivalent.[2] Hence, it suffices to show that $\mathbf{R}F^*$ is equivalent to $(\mathbf{R}F)^*$. Since the total derived functors of adjoint homotopical functors are again adjoint,[3] $\mathbf{L}F_!$ is left adjoint to $\mathbf{R}F^*$. As the adjoints of a given functor are unique (up to equivalence),[4] we can equivalently show that $\mathbf{L}F_!$ is left adjoint to $(\mathbf{R}F)^*$, i.e. it is equivalent to $(\mathbf{R}F)_!$. Using the universal property of Kan extensions, we may compute the following:

$$\begin{aligned}
& \mathbf{L}F_! \rho_{C,wC} \exists_{LC} \simeq \rho_{D,wD} F_! \exists_{LC} \\
\text{and} \quad & (\mathbf{R}F)_! \text{よ}_{LC} \gamma_C \simeq \text{よ}_{LD} \mathbf{R}F \gamma_C \simeq \text{よ}_{LD} \gamma_D F.
\end{aligned}$$

Recall that $\exists_{LC}$ has been defined to be the composition of $\gamma^\flat_{C,!}$ and $\exists_C$. As $\gamma^\flat_{C,!}$ is the identity of $\operatorname{Fun}(C, \mathrm{sSet})$ on the underlying category, $\exists_{LC}$ and $\exists_C$ are the same functor on the underlying categories. By the preceding paragraph, we have shown $F_! \exists_C \simeq \exists_D F$ and, thus, $\mathbf{L}F_! \rho_{C,wC} \exists_{LC}$ is equivalent to $\rho_{D,wD} \exists_D F$ (on the underlying categories). By the formula in (4.18.1), $\rho_{D,wD} \exists_D$ and $\text{よ}_{LD} \gamma_D$ are equivalent, so that $\mathbf{L}F_! \rho_{C,wC} \exists_{LC}$ and $(\mathbf{R}F)_! \text{よ}_{LC} \gamma_C$ are equivalent, namely to $\text{よ}_{LD} \gamma_D F$. The same formula also implies that $\mathbf{L}F_! \rho_{C,wC} \exists_{LC}$ is equivalent to $\mathbf{L}F_! \text{よ}_{LC} \gamma_C$. Therefore, by a combination of the universal properties of よ_{LC} and γ_C, we have that $\mathbf{L}F_!$ and $(\mathbf{R}F)_!$ are equivalent.

In total, we have constructed a diagram in FibCat

$$\begin{tikzcd}
P_{wC}(C)_{\mathrm{wrf}} \arrow[d, hook] & P_{wD}(D)_{\mathrm{wrf}} \arrow[l, "F^*"'] \arrow[d, hook] \\
P_{wC}(C)_{\mathrm{wrep}} & P_{wD}(D)_{\mathrm{wrep}} \arrow[l, "F^*"'] \\
C \arrow[u, "\exists_{LC}"] \arrow[r, "F"] & D \arrow[u, "\exists_{LD}"]
\end{tikzcd}$$

[1] See [Cis19, Proposition 6.1.6].

[2] See [Cis19, Theorem 6.1.23].

[3] See [Cis19, Proposition 7.1.14] or, in our case, [Cis19, Theorem 7.5.30].

[4] See [Cis19, Proposition 6.1.9].

which does not necessarily commute as is, but up to homotopy, that is, it commutes after passing to total right derived functors. This means that the equivalence $\mathbf{R}F\colon LC \to LD$ is the image of a zig-zag of (strictly) left exact functors under $\bar{L}\colon L\mathrm{FibCat}_f \to \mathrm{LexCat}_\infty$. By dualization, this shows how to replace homotopically right exact functors by a zig-zag of (strictly) right exact functors.

Remark. Note that $F^*\colon P_{wD}(D)_{\mathrm{wrep}} \to P_{wC}(C)_{\mathrm{wrep}}$ is not left exact as right Quillen functor do not preserve weak equivalences between all objects, but only the fibrant ones.

Let C and D be homotopical categories of cofibrant objects. Given an equivalence $\mathcal{F}\colon LC \to LD$, the following paragraph concludes the rectification problem by exhibiting $\mathcal{F}$ as a total derived functor of some homotopical functor $F\colon C \to D$.

4.23. Since $\exists_{LD^{\mathrm{op}}}\colon D^{\mathrm{op}} \to P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})$ is a weak equivalence in FibCat by Lemma 4.16, which factors through $P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}}$ such that the resulting functor $\exists_{LD^{\mathrm{op}}}\colon D^{\mathrm{op}} \to P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}}$ is still a weak equivalence by the observations in Lemma 4.21. We may assume without loss of generality that D is of the form $(P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}})^{\mathrm{op}}$. Since the Yoneda embedding induces an equivalence of LD and $\mathrm{Fun}(LD^{\mathrm{op}}, \mathcal{S})_{\mathrm{rep}}$, we would like to recall that there is a localization functor of the form $P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}} \to LD$, which we denote by $\rho_{D^{\mathrm{op}},wD^{\mathrm{op}}}$. By paragraph Lemma 4.22 together with Lemma 4.12, it suffices to construct a (homotopically right exact) functor

$$F\colon C \longrightarrow (P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}})^{\mathrm{op}}$$

such that

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma_C"'] & (P_{wD^{\mathrm{op}}}(D^{\mathrm{op}})_{\mathrm{wrep}})^{\mathrm{op}} \arrow[d, "\rho_{D^{\mathrm{op}},wD^{\mathrm{op}}}"] \\
LC \arrow[r, "\mathcal{F}"'] & LD
\end{tikzcd}$$

is a commutative square of ∞-categories. By (the dual of) Lemma 4.20 together with the remarks of Lemma 4.18, the opposite of such a functor F corresponds to a commutative diagram of ∞-categories the form

$$\begin{tikzcd}
C^{\mathrm{op}} \arrow[r] \arrow[d, "\mathcal{F}^{\mathrm{op}}\gamma_C^{\mathrm{op}}"'] & P(D^{\mathrm{op}}) \arrow[d, "\rho_{D^{\mathrm{op}}}"] \\
LD^{\mathrm{op}} \arrow[r, "\gamma^*_{D^{\mathrm{op}}}\text{よ}_{LD^{\mathrm{op}}}"'] & \mathrm{Fun}((D^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S}).
\end{tikzcd}$$

Recall that $\gamma_C^{\mathrm{op}}\colon C^{\mathrm{op}} \to LC^{\mathrm{op}}$ is a localization functor[1] and, thus, we would like to denote γ_C^{op} by $\gamma_{C^{\mathrm{op}}}$. In particular, $\gamma^*_{C^{\mathrm{op}}}$ is fully faithful, so that $\gamma_{C^{\mathrm{op}},!}\gamma^*_{C^{\mathrm{op}}}$ is equivalent to the identity. Hence, the diagram

$$\begin{tikzcd}
LC^{\mathrm{op}} \arrow[r, "\gamma^*_{C^{\mathrm{op}}}\text{よ}_{LC^{\mathrm{op}}}"] \arrow[d, "\mathcal{F}^{\mathrm{op}}"'] & \mathrm{Fun}((C^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S}) \arrow[d, "\gamma^*_{D^{\mathrm{op}}}\mathcal{F}^{\mathrm{op}}_!\gamma_{C^{\mathrm{op}},!}"] \\
LD^{\mathrm{op}} \arrow[r, "\gamma^*_{D^{\mathrm{op}}}\text{よ}_{LD^{\mathrm{op}}}"'] & \mathrm{Fun}((D^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

[1] See [Cis19, Proposition 7.1.7].

commutes. Let

$$^{t}\left(\gamma^*_{D^{\mathrm{op}}}\mathcal{F}_!^{\mathrm{op}}\gamma_{C^{\mathrm{op}},!}\text{よ}_{C^{\mathrm{op}}}\right)\colon C^{\mathrm{op}} \times \left(D^{\mathrm{op}}\right)^{\mathrm{op}} \longrightarrow \mathcal{S}$$

denote the transposed functor. Because there is a localization functor of the form $\mathrm{Fun}\left(C^{\mathrm{op}} \times \left(D^{\mathrm{op}}\right)^{\mathrm{op}}, \mathrm{sSet}\right) \to \mathrm{Fun}\left(C^{\mathrm{op}} \times \left(D^{\mathrm{op}}\right)^{\mathrm{op}}, \mathcal{S}\right)$ given by postcomposing with the localization $\mathrm{sSet}_{\mathrm{KQ}} \to \mathcal{S}$,[1] there is a functor $f\colon C^{\mathrm{op}} \times \left(D^{\mathrm{op}}\right)^{\mathrm{op}} \to \mathrm{sSet}$ which becomes naturally isomorphic to $^{t}\left(\gamma^*_{D^{\mathrm{op}}}\mathcal{F}_!^{\mathrm{op}}\gamma_{C^{\mathrm{op}},!}\text{よ}_{C^{\mathrm{op}}}\right)$ after postcomposing with $\mathrm{sSet}_{\mathrm{KQ}} \to \mathcal{S}$. Hence, if $\hat{f}\colon C^{\mathrm{op}} \to P\left(D^{\mathrm{op}}\right)$ denotes the curried functor, then the diagram of ∞-categories

$$\begin{tikzcd}
 & P\left(D^{\mathrm{op}}\right) \arrow[dr, "\rho_{D^{\mathrm{op}}}"] & \\
C^{\mathrm{op}} \arrow[ur, "\hat{f}"] \arrow[rr, "\gamma^*_{D^{\mathrm{op}}}\mathcal{F}_!^{\mathrm{op}}\gamma_{C^{\mathrm{op}},!}\text{よ}_{C^{\mathrm{op}}}"'] & & \mathrm{Fun}\left(\left(D^{\mathrm{op}}\right)^{\mathrm{op}}, \mathcal{S}\right)
\end{tikzcd}$$

commutes. Recall here that $\rho_{D^{\mathrm{op}}}$ itself has been defined by postcomposing with $\mathrm{sSet}_{\mathrm{KQ}} \to \mathcal{S}$. By the universal property of colimit extensions, we may compute

$$\gamma^*_{D^{\mathrm{op}}}\mathcal{F}_!^{\mathrm{op}}\gamma_{C^{\mathrm{op}},!}\text{よ}_{C^{\mathrm{op}}} \simeq \gamma^*_{D^{\mathrm{op}}}\mathcal{F}_!^{\mathrm{op}}\text{よ}_{LC^{\mathrm{op}}}\gamma_{C^{\mathrm{op}}} \simeq \gamma^*_{D^{\mathrm{op}}}\text{よ}_{LD^{\mathrm{op}}}\mathcal{F}^{\mathrm{op}}\gamma_{C^{\mathrm{op}}},$$

so that the diagram

$$\begin{tikzcd}
C^{\mathrm{op}} \arrow[r, "\hat{f}"] \arrow[d, "\mathcal{F}^{\mathrm{op}}\gamma_{C^{\mathrm{op}}}"'] & P\left(D^{\mathrm{op}}\right) \arrow[d, "\rho_{D^{\mathrm{op}}}"] \\
LD^{\mathrm{op}} \arrow[r, "\gamma^*_{D^{\mathrm{op}}}\text{よ}_{LD^{\mathrm{op}}}"'] & \mathrm{Fun}\left(\left(D^{\mathrm{op}}\right)^{\mathrm{op}}, \mathcal{S}\right)
\end{tikzcd}$$

commutes per construction. This gives us (the opposite of) the desired functor F.

With this, we can conclude:

Theorem 4.24. *$L\colon \mathrm{CofCat}_f \to \mathrm{RexQCat}$ is a weak equivalence of categories with weak equivalences and fibrations.*

Corollary 4.25. *All morphisms in the diagram of categories with weak equivalences and fibrations*

$$\begin{tikzcd}
\mathrm{CofCat}_f \arrow[r, hook, shift right, "N"'] \arrow[d, shift right] & \mathrm{CofQCat}_f \arrow[l, shift right, "\mathrm{ho}"'] \arrow[r, shift left, "L'"] \arrow[d, shift right] & \mathrm{RexQCat} \arrow[l, hook, shift left, "i'"] \\
\mathrm{CofCat} \arrow[u, shift right, "\ell"'] \arrow[r, hook, shift right, "N"'] & \mathrm{CofQCat} \arrow[l, shift right, "\mathrm{ho}"'] \arrow[u, shift right, "\ell"'] &
\end{tikzcd}$$

are homotopical, and each pair induces total derived functors, which are mutually inverse to each other and thus, in particular, equivalences of ∞-categories.

[1] See [Cis19, Theorem 7.9.8].

A Appendix

The Construction of a Path Object

Let C be a homotopical quasi-category of cofibrant objects. Lemma 2.31 is going to be proven in the following manner:

1. By Lemma A.2, $\operatorname{path}(C)$ is a homotopical quasi-category of cofibrant objects.
2. By Lemma A.3, the Λ^2_2-ary diagonal functor $\Delta_{\Lambda^2_2}\colon C \to \operatorname{path}(C)$ and the evaluation functor $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \operatorname{path}(C) \to C \times C$ are right exact.
3. By Lemma A.4, the Λ^2_2-ary diagonal is a weak equivalence in CofQCat.
4. By Lemma A.5, the evaluation functor is a fibration in CofQCat.
5. By Lemma A.6, $\operatorname{path}(C)$ together with $\Delta_{\Lambda^2_2}$ and $(\mathrm{ev}_0, \mathrm{ev}_1)$ is a factorization of the ordinary diagonal of C.

Notation A.1. Since Λ^2_2 is (equivalent to) the nerve of the partially ordered set of non-empty subsets of $\{0, 1\}$, we would like to denote objects a in $\operatorname{path}(C)$ as spans in C of the form

$$a_0 \xrightarrow{a_{0\subseteq 01}} a_{01} \xleftarrow{a_{1\subseteq 01}} a_1$$

with $a_{0\subseteq 01}$ and $a_{1\subseteq 01}$ trivial cofibrations in C, and maps $f\colon a \to b$ in $\operatorname{path}(C)$ as commutative diagram in C of the form

$$\begin{tikzcd} a_0 \arrow[r, "a_{0\subseteq 01}"] \arrow[d, "f_0"] & a_{01} \arrow[d, "f_{01}"] & a_1 \arrow[l, "a_{1\subseteq 01}"'] \arrow[d, "f_1"] \\ b_0 \arrow[r, "b_{0\subseteq 01}"'] & b_{01} & b_1. \arrow[l, "b_{1\subseteq 01}"] \end{tikzcd}$$

Proposition A.2. *[Szu14, Proposition 1.13]* $\operatorname{path}(C)$ *together with the classes of maps as defined in Lemma 2.30 is a homotopical quasi-category of cofibrant objects.*

Proof. $\operatorname{path}(C)$ admits an initial object: As colimits in functor quasi-categories are computed componentwise, the constant span

$$\varnothing_C \xrightarrow{1_{\varnothing_C}} \varnothing_C \xleftarrow{1_{\varnothing_C}} \varnothing_C$$

is an initial object in $\operatorname{Fun}\left(\Lambda^2_2, C\right)$. Since $1_{\varnothing_C}$ is a trivial cofibration in C, this span is an object in $\operatorname{path}(C)$, thus initial in $\operatorname{path}(C)$.

All objects in $\operatorname{path}(C)$ are cofibrant: Let a be an object in $\operatorname{path}(C)$ and consider the canonical map $\varnothing_{\operatorname{path}(C)} \to a$ given by the diagram in C

$$\begin{tikzcd} \varnothing_C \arrow[r] \arrow[d] & \varnothing_C \arrow[d] & \varnothing_C \arrow[l] \arrow[d] \\ a_0 \arrow[r, "a_{0\subseteq 01}"'] & a_{01} & a_1, \arrow[l, "a_{1\subseteq 01}"] \end{tikzcd}$$

which commutes by intiality of $\varnothing_C$. Because there is a canonical isomorphism $a_\epsilon \sqcup \varnothing_C \to a_\epsilon$ for each $\epsilon \in \{0,1\}$, the induced map $a_\epsilon \sqcup \varnothing_C \to a_{01}$ is, as the composition of $a_\epsilon \sqcup \varnothing_C \to a_\epsilon$ and $a_{\epsilon\subseteq 01}$, a cofibration. Since all objects in C are cofibrant, the component maps $\varnothing_C \to a_\epsilon$ are cofibrations, and thus $\varnothing_{\mathrm{path}(C)} \to a$ is a cofibration, i.e. a is cofibrant in $\mathrm{path}(C)$.

The class of weak equivalences in $\mathrm{path}(C)$ satisfies the 2-out-of-6 property as the class of weak equivalences in C satisfies this property and weak equivalences in $\mathrm{path}(C)$ are defined componentwise. Due to this componentwise definition, for any weak equivalence $a \to b$ in $\mathrm{path}(C)$, the identities of a and b are weak equivalences as well.

The class of cofibrations in $\mathrm{path}(C)$ contains all identities: Let a be an object in $\mathrm{path}(C)$. Clearly, all component maps of 1_a are identities, thus cofibrations in C. It remains to show that the induced map $(1_{a_{01}}, a_{\epsilon\subseteq 01}) : a_{01} \sqcup_{a_\epsilon} a_\epsilon \to a_{01}$ is a cofibration for all $\epsilon \in \{0,1\}$. As the left inner square and the outer rectangle in this commutative diagram in C

$$\begin{tikzcd}
a_\epsilon \arrow[r, "1_{a_\epsilon}"] \arrow[d, "a_{\epsilon\subseteq 01}"] & a_\epsilon \arrow[r, "1_{a_\epsilon}"] \arrow[d] & a_\epsilon \arrow[d, "a_{\epsilon\subseteq 01}"] \\
a_{01} \arrow[r] & a_{01} \sqcup_{a_\epsilon} a_\epsilon \arrow[r, "(1_{a_{01}}, a_{\epsilon\subseteq 01})"'] & a_{01}
\end{tikzcd}$$

are pushout square, so is the right inner square by the pasting law. As 1_{a_ϵ} is a (trivial) cofibration, so is $(1_{a_{01}}, a_{\epsilon\subseteq 01})$ by the pushout axiom.[1]

The class of cofibrations is closed under composition: let $i\colon a \to b$ and $j\colon b \to c$ be cofibrations in $\mathrm{path}(C)$. For each $\epsilon \in \{0,1\}$, the component map $(q \circ p)_\epsilon\colon a_\epsilon \to c_\epsilon$ is a composition of cofibrations in C, thus itself a cofibration, and it remains to show that the induced map $((q \circ p)_{01}, c_{\epsilon\subseteq 01}) : a_{01} \sqcup_{a_\epsilon} c_\epsilon \to c_{01}$ is a cofibration in C for each $\epsilon \in \{0,1\}$. By the pasting law, each square and rectangle in the following diagram in C

$$\begin{tikzcd}
a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "p_\epsilon"] & a_{01} \arrow[d] & \\
b_\epsilon \arrow[r] \arrow[d, "q_\epsilon"] & a_{01} \sqcup_{a_\epsilon} b_\epsilon \arrow[r, "(p_{01}, b_{\epsilon\subseteq 01})"] \arrow[d, "1_{a_{01}} \sqcup q_\epsilon"] & b_{01} \arrow[d] \\
c_\epsilon \arrow[r] & a_{01} \sqcup_{a_\epsilon} c_\epsilon \arrow[r, "p_{01} \sqcup 1_{c_\epsilon}"] & b_{01} \sqcup_{b_\epsilon} c_\epsilon
\end{tikzcd}$$

is pushout for each $\epsilon \in \{0,1\}$. As $(p_{01}, b_{\epsilon\subseteq 01})$ is a cofibration by assumption, so is $p_{01} \sqcup 1_{c_\epsilon}$ by the pushout axiom. Moreover, $(q_{01}, c_{\epsilon\subseteq 01}) : b_{01} \sqcup_{b_\epsilon} c_\epsilon \to c_{01}$ is assumed to be a cofibration, so that $((q \circ p)_{01}, c_{0\subseteq 01}) : a_{01} \sqcup_{a_\epsilon} c_\epsilon \to c_{01}$, which is a composition of $p_{01} \sqcup 1_{c_\epsilon}$ and $(q_{01}, c_{\epsilon\subseteq 01})$, is a cofibration as well.

The class of (trivial) cofibrations satisfies the pushout axiom: Let

$$b \xleftarrow{\ i\ } a \xrightarrow{\ f\ } a'$$

be a span in $\mathrm{path}(C)$ with i a (trivial) cofibration. Since it is componentwise a span of trivial cofibrations in C, its pushout exists in $\mathrm{Fun}\left(\Lambda^2_2, C\right)$ because

[1] Alternatively, one can show that $a_{01} \sqcup_{a_\epsilon} a_\epsilon \to a_{01}$ is an isomorphism of cofibrant objects.

of the pushout axiom in C and because colimits in functor quasi-categories are computed componentwise. Denote this pushout in $\mathrm{Fun}\left(\Lambda^2_2, C\right)$ as follows:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & a' \arrow[d, "i'"] \\ b \arrow[r, "f'"'] & b'. \end{tikzcd}$$

Hence, for $\eta \in \{0, 1, 01\}$, the induced diagram in C

$$\begin{tikzcd} a_\eta \arrow[r, "f_\eta"] \arrow[d, "i_\eta"'] & a'_\eta \arrow[d, "i'_\eta"] \\ b_\eta \arrow[r, "f'_\eta"'] & b'_\eta \end{tikzcd}$$

is a pushout square, so that i'_η is a (trivial) cofibration because of the pushout axiom and because (trivial) cofibrations in $\mathrm{path}(C)$ are componentwise such. We want to show that b' belongs to $\mathrm{path}(C)$, i.e. the component map $b'_{\epsilon\subseteq 01} : b'_\epsilon \to b'_{01}$ for $\epsilon \in \{0, 1\}$ is a trivial cofibration. Note that it is given by the commutative diagram in C

$$\begin{tikzcd} b_\epsilon \arrow[d, "b_{\epsilon\subseteq 01}"] & a_\epsilon \arrow[l, hook', "i_\epsilon"'] \arrow[r, "f_\epsilon"] \arrow[d, "a_{\epsilon\subseteq 01}"] & a'_\epsilon \arrow[d, "a'_{\epsilon\subseteq 01}"] \\ b_{01} & a_{01} \arrow[l, hook', "i_{01}"] \arrow[r, "f'_{01}"'] & a'_{01} \end{tikzcd}$$

such that the following diagrams in C

$$\begin{tikzcd} a'_\epsilon \arrow[r, "i'_\epsilon"] \arrow[d, "a'_{\epsilon\subseteq 01}"'] & b'_\epsilon \arrow[d, "b'_{\epsilon\subseteq 01}"] \\ a'_{01} \arrow[r, "i'_{01}"'] & b'_{01} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b_\epsilon \arrow[r, "f'_\epsilon"] \arrow[d, "b_{\epsilon\subseteq 01}"'] & b'_\epsilon \arrow[d, "b'_{\epsilon\subseteq 01}"] \\ b_{01} \arrow[r, "f'_{01}"'] & b'_{01} \end{tikzcd}$$

commute. Now, choose pushout squares in C of the following form

$$\begin{tikzcd} a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"'] & a_{01} \arrow[d, "j_\epsilon"] \\ b_\epsilon \arrow[r, "s_\epsilon"'] & x_\epsilon \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a'_\epsilon \arrow[r, "a'_{\epsilon\subseteq 01}"] \arrow[d, "i'_\epsilon"'] & a'_{01} \arrow[d, "j'_\epsilon"] \\ b'_\epsilon \arrow[r, "s'_\epsilon"'] & y_\epsilon \end{tikzcd}$$

such that j_ϵ and j'_ϵ are (trivial) cofibrations and s_ϵ and s'_ϵ are trivial cofibrations, by the pushout axiom. By the pasting law, the outer reactangle of the commutative diagram in C

$$\begin{tikzcd} a_\epsilon \arrow[r, "f_\epsilon"] \arrow[d, "i_\epsilon"] & a'_\epsilon \arrow[r, "a'_{\epsilon\subseteq 01}"] \arrow[d, "i'_\epsilon"] & a'_{01} \arrow[d, "j'_\epsilon"] \\ b_\epsilon \arrow[r, "f'_\epsilon"'] & b'_\epsilon \arrow[r, "s'_\epsilon"'] & y_\epsilon \end{tikzcd} \tag{A.2.1}$$

is a pushout square, so that the commutative diagram in C

$$\begin{tikzcd}
b_\epsilon \arrow[d, "1_{b_\epsilon}"] & a_\epsilon \arrow[l, "i_\epsilon"'] \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "1_{a_\epsilon}"] & a_{01} \arrow[d, "f_{01}"] \\
b_\epsilon & a_\epsilon \arrow[l, "i_\epsilon"] \arrow[r, "a'_{\epsilon\subseteq 01} f_\epsilon"'] & a'_{01}
\end{tikzcd}$$

induces a map $1_{b_\epsilon} \sqcup f_{01} \colon x_\epsilon \to y_\epsilon$ such that the diagrams in C

$$\begin{tikzcd}
b_\epsilon \arrow[r, "s_\epsilon"] \arrow[d, "f'_\epsilon"'] & x_\epsilon \arrow[d, "1_{b_\epsilon}\sqcup f_{01}"] \\
b'_\epsilon \arrow[r, "s'_\epsilon"'] & y_\epsilon
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a_{01} \arrow[r, "j_\epsilon"] \arrow[d, "f_{01}"'] & x_\epsilon \arrow[d, "1_{b_\epsilon}\sqcup f_{01}"] \\
a'_{01} \arrow[r, "j'_\epsilon"'] & y_\epsilon
\end{tikzcd}$$

commute. In particular, the diagram in C

$$\begin{tikzcd}
a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"] & a_{01} \arrow[r, "f_{01}"] \arrow[d, "j_\epsilon"] & a'_{01} \arrow[d, "j'_\epsilon"] \\
b_\epsilon \arrow[r, "s_\epsilon"'] & x_\epsilon \arrow[r, "1_{b_\epsilon}\sqcup f_{01}"'] & y_\epsilon
\end{tikzcd} \tag{A.2.2}$$

commutes, and one can check that the outer rectangle is the outer rectangle in (A.2.1), which is a pushout square. Hence, by the pasting law, the right inner square in (A.2.2) is a pushout square. Now, the commutative square in C

$$\begin{tikzcd}
a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"'] & a_{01} \arrow[d, "i_{01}"] \\
b_\epsilon \arrow[r, "b_{\epsilon\subseteq 01}"'] & b_{01}
\end{tikzcd}$$

induces a map $(b_{\epsilon\subseteq 01}, i_{01}) : x_\epsilon \to b_{01}$ such that the triangles in C

$$\begin{tikzcd}
 & x_\epsilon \arrow[dr, "(b_{\epsilon\subseteq 01}, i_{01})"] & \\
b_\epsilon \arrow[ur, "s_\epsilon"] \arrow[rr, "b_{\epsilon\subseteq 01}"'] & & b_{01}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & x_\epsilon \arrow[dr, "(b_{\epsilon\subseteq 01}, i_{01})"] & \\
a_{01} \arrow[ur, "j_\epsilon"] \arrow[rr, "i_{01}"'] & & b_{01}
\end{tikzcd}$$

commute. Due to the commutativity of the left triangle, $(b_{\epsilon\subseteq 01}, i_{01})$ is a weak equivalence by the 2-out-of-3 property. Analogously, the commutative square in C

$$\begin{tikzcd}
a'_\epsilon \arrow[r, "a'_{\epsilon\subseteq 01}"] \arrow[d, "i'_\epsilon"'] & a'_{01} \arrow[d, "i'_{01}"] \\
b'_\epsilon \arrow[r, "b'_{\epsilon\subseteq 01}"'] & b'_{01}
\end{tikzcd}$$

induces a map $\left(b'_{\epsilon\subseteq 01}, i'_{01}\right) : y_\epsilon \to b'_{01}$ such that the triangles in C

$$\begin{tikzcd}
 & x_\epsilon \arrow[dr, "(b'_{\epsilon\subseteq 01}, i'_{01})"] & \\
b'_\epsilon \arrow[ur, "s_\epsilon"] \arrow[rr, "b'_{\epsilon\subseteq 01}"'] & & b'_{01}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & x_\epsilon \arrow[dr, "(b'_{\epsilon\subseteq 01}, i'_{01})"] & \\
a'_{01} \arrow[ur, "j_\epsilon"] \arrow[rr, "i'_{01}"'] & & b'_{01}
\end{tikzcd}$$

commute. With this, one can check that this diagram in C

$$\begin{tikzcd}
a_{01} \arrow[r, "j_\epsilon"] \arrow[d, "f_{01}"] & x_\epsilon \arrow[r, "(b_{\epsilon\subseteq 01}, i_{01})"] \arrow[d, "1_{b_\epsilon} \sqcup f_{01}"] & b_{01} \arrow[d, "f'_{01}"] \\
a'_{01} \arrow[r, "j'_\epsilon"'] & y_\epsilon \arrow[r, "(b'_{\epsilon\subseteq 01}, i'_{01})"'] & b'_{01}
\end{tikzcd}$$

commutes, so that the right inner square is a pushout square by the pasting law. Because i is a cofibration in $\mathrm{path}(C)$, the map $(b_{\epsilon\subseteq 01}, i_{01})$ is a cofibration, thus a trivial cofibration. By the pushout axiom, $\left(b'_{\epsilon\subseteq 01}, i'_{01}\right)$ is a trivial cofibration. As a composition of s'_ϵ and $\left(b'_{\epsilon\subseteq 01}, i'_{01}\right)$, the map b'_{01} is a trivial cofibration. Therefore, b' belongs to $\mathrm{path}(C)$ and we have also shown that i' is a (trivial) cofibration. Moreover, as $\mathrm{path}(C)$ is a full subcategory of $\mathrm{Fun}\left(\Lambda^2_2, C\right)$, any pushout of the span in $\mathrm{path}(C)$

$$b \xleftarrow{i} a \xrightarrow{f} a'$$

is of the form above, so that the proof that i' is a (trivial) cofibration holds for any pushout square in $\mathrm{path}(C)$ of the form

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "i"'] & a' \arrow[d, "i'"] \\
b \arrow[r, "f'"'] & b'
\end{tikzcd}$$

in which i is a (trivial) cofibration.

Lastly, any map in $\mathrm{path}(C)$ (with cofibrant source) factors into a cofibration followed by a weak equivalence: Let $f\colon a \to b$ be a map in $\mathrm{path}(C)$, i.e. a commutative diagram in C of the form

$$\begin{tikzcd}
a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "f_\epsilon"'] & a_{01} \arrow[d, "f_{01}"] \\
b_\epsilon \arrow[r, "b_{\epsilon\subseteq 01}"'] & b_{01}
\end{tikzcd}$$

for each $\epsilon \in \{0, 1\}$. Now, choose a factorization in C of the form

$$\begin{tikzcd}
 & b'_\epsilon \arrow[dr, "s_\epsilon"] & \\
a_\epsilon \arrow[ur, "i_\epsilon"] \arrow[rr, "f_\epsilon"'] & & b_\epsilon
\end{tikzcd}$$

with i_ϵ a cofibration and s_ϵ a weak equivalence in C, and choose a pushout square in C

$$\begin{tikzcd}
a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"'] & a_{01} \arrow[d, "\iota_2^\epsilon"] \\
b'_\epsilon \arrow[r, "\iota_1^\epsilon"'] & x_\epsilon
\end{tikzcd}$$

Then this diagram

$$\begin{tikzcd} a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"'] & a_{01} \arrow[d, "f_{01}"] \\ b'_\epsilon \arrow[r, "b_{\epsilon\subseteq 01} s_\epsilon"'] & b_{01} \end{tikzcd}$$

commutes and induces a map $g_\epsilon\colon x_\epsilon \to b_{01}$ with $g_\epsilon \iota_1^\epsilon = b_{\epsilon\subseteq 01} s_\epsilon$ and $g_\epsilon \iota_2^\epsilon = f_{01}$. Again, choose a pushout square in C

$$\begin{tikzcd} a_{01} \arrow[r, "\iota_2^1"] \arrow[d, "\iota_2^0"'] & x_1 \arrow[d, "\kappa_1"] \\ x_0 \arrow[r, "\kappa_0"'] & x \end{tikzcd}$$

where κ_0 and κ_1 are cofibrations. (Note the unfortunate, but unavoidable switch of indices.) Then this diagram

$$\begin{tikzcd} a_{01} \arrow[r, "\iota_2^1"] \arrow[d, "\iota_2^0"'] & x_1 \arrow[d, "g_1"] \\ x_0 \arrow[r, "g_0"'] & b_{01} \end{tikzcd}$$

commutes, as the diagonal is f_{01}, and induces a map $g\colon x \to b_{01}$ such that $g\kappa_\epsilon = g_\epsilon$. Lastly, choose a factorization of the form

$$\begin{tikzcd} & b'_{01} \arrow[dr, "s_{01}"] & \\ x \arrow[ur, "j"] \arrow[rr, "g"'] & & b_{01} \end{tikzcd}$$

with j a cofibration and s_{01} a weak equivalence in C. Per construction, this diagram in C

$$\begin{tikzcd} a_\epsilon \arrow[r, "a_{\epsilon\subseteq 01}"] \arrow[d, "i_\epsilon"'] & a_{01} \arrow[d, "j\kappa_\epsilon\iota_2^\epsilon"] \\ b'_\epsilon \arrow[r, "j\kappa_\epsilon\iota_1^\epsilon"] \arrow[d, "s_\epsilon"'] & b'_{01} \arrow[d, "s_{01}"] \\ b_\epsilon \arrow[r, "b_{\epsilon\subseteq 01}"] & b_{01} \end{tikzcd} \tag{A.2.3}$$

commutes: for the upper square, use the formula $\iota_2^\epsilon a_{\epsilon\subseteq 01} = \iota_1^\epsilon i_\epsilon$; for the lower square, use

$$s_{01} j\kappa_\epsilon \iota_1^\epsilon = g\kappa_\epsilon \iota_1^\epsilon = g_\epsilon \iota_1^\epsilon = b_{\epsilon\subseteq 01} s_\epsilon.$$

It remains to show that $b'_{\epsilon\subseteq 01} := j\kappa_\epsilon\iota_1^\epsilon$ is a trivial cofibration in C. In fact, it is the composition of cofibrations, and $b'_{\epsilon\subseteq 01}$ is a weak equivalence by the 2-out-of-3 property as the lower square in (A.2.3) commutes. This gives us an object b' in $\mathrm{path}(C)$ and, once we set $i_{01} := j\kappa_\epsilon\iota_2^\epsilon$, we get a factorization of the form

$$\begin{tikzcd} & c \arrow[dr, "s"] & \\ a \arrow[ur, "i"] \arrow[rr, "f"'] & & b \end{tikzcd}$$

where s is obviously a weak equivalences and i is a cofibration because, per construction, i_ϵ is a cofibration in C, and any pushout $a_{01} \sqcup_{a_\epsilon} b'_\epsilon$ is canonically isomorphic to x_ϵ and the canonical map $x_\epsilon \to b'_{01}$ is the composition of the cofibrations κ_ϵ and j. □

Proposition A.3. *[Szu14, Theorem 1.14] The Λ^2_2-ary diagonal $\Delta_{\Lambda^2_2}\colon C \to \mathrm{path}(C)$ and the evaluation functor $(\mathrm{ev}_0, \mathrm{ev}_1) : \mathrm{path}(C) \to C \times C$ are right exact.*

Remark. Since all identities in C are trivial cofibrations, constant spans in C are objects in $\mathrm{path}(C)$, so that, in fact, the Λ^2_2-ary diagonal functor factors through $\mathrm{path}(C)$.

Proof of lemma. As limits and colimits in functor categories (or fully faithfully embedded subcategories of such) are computed componentwise, $\Delta_{\Lambda^2_2}$ and $(\mathrm{ev}_0, \mathrm{ev}_1)$ preserve initial objects and pushouts of cofibrations. Let $f\colon a \to b$ be a (trivial) cofibration in C. Then $\Delta_{\Lambda^2_2}(f)$ is the commutative diagram in C

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "f"] & a \arrow[d, "f"] & a \arrow[l, "1_a"'] \arrow[d, "f"] \\ b \arrow[r, "1_b"'] & b & b \arrow[l, "1_b"] \end{tikzcd}$$

and clearly componentwise a (trivial) cofibration. Moreover, the induced map $a \sqcup_a b \to b$ is an isomorphism of cofibrant objects, thus a cofibration. Therefore, $\Delta_{\Lambda^2_2}$ preserves cofibrations and trivial cofibrations. Lastly, as weak equivalences and cofibrations in $\mathrm{path}(C)$ are, in particular, componentwise, $(\mathrm{ev}_0, \mathrm{ev}_1)$ preserves cofibrations and trivial cofibrations. □

Proposition A.4. *The Λ^2_2-ary diagonal functor $\Delta_{\Lambda^2_2}\colon C \to \mathrm{path}(C)$ is a weak equivalence in* CofQCat.

Proof. We verify the condition of Lemma 2.22 citing [Cis19, Proposition 7.6.15] for $\Delta_{\Lambda^2_2}$. Let $f\colon a \to b$ be a map in C such that $\Delta_{\Lambda^2_2}(f)$ is a weak equivalence in $\mathrm{path}(C)$, i.e. all vertical maps in the following commutative diagram in C are weak equivalences:

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "f"] & a \arrow[d, "f"] & a \arrow[l, "1_a"'] \arrow[d, "f"] \\ b \arrow[r, "1_b"'] & b & b. \arrow[l, "1_b"] \end{tikzcd}$$

This means that f is already a weak equivalence in C. Now, let $g\colon \Delta_{\Lambda^2_2}(c) \to d$ be a map in $\mathrm{path}(C)$, i.e. a commutative diagram in C of the form

$$\begin{tikzcd} c \arrow[r, "1_c"] \arrow[d, "g_0"] & c \arrow[d, "g_{01}"] & c \arrow[l, "1_c"'] \arrow[d, "g_1"] \\ d_0 \arrow[r, "s_0"'] & d_{01} & d_1. \arrow[l, "s_1"] \end{tikzcd}$$

Then note that the composites $s_0 g_0$ and $s_1 g_1$ are equal[1], namely both equal to g_{01}. Moreover, the following diagram in C

$$\begin{tikzcd} d_0 \arrow[r, "s_0"] \arrow[d, "s_0"] & d_{01} \arrow[d, "1_{d_{01}}"] & d_1 \arrow[l, "s_1"'] \arrow[d, "s_1"] \\ d_{01} \arrow[r, "1_{d_{01}}"'] & d_{01} & d_{01} \arrow[l, "1_{d_{01}}"] \end{tikzcd}$$

commutes and defines a weak equivalence $s\colon d \to \Delta_{\Lambda^2_2}(d_{01})$ in path(C). Then the composite $sg\colon \Delta_{\Lambda^2_2}(c) \to \Delta_{\Lambda^2_2}(d_{01})$ is componentwise g_{01}, so that the following diagram in path(C) commutes:

$$\begin{tikzcd} \Delta_{\Lambda^2_2}(c) \arrow[r, "g"] \arrow[d, "\Delta_{\Lambda^2_2}(g_{01})"'] & d \arrow[d, "s"] \\ \Delta_{\Lambda^2_2}(d_{01}) \arrow[r, "1_{\Delta_{\Lambda^2_2}(d_{01})}"'] & \Delta_{\Lambda^2_2}(d_{01}). \end{tikzcd}$$

□

Proposition A.5. *The evaluation functor* $(\mathrm{ev}_0, \mathrm{ev}_1) : \mathrm{path}(C) \to C \times C$ *is a fibration in* CofQCat.

Proof. $(\mathrm{ev}_0, \mathrm{ev}_1)$ is an inner fibration: Let k and n be natural numbers such that $0 < k < n$ and denote the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ by i. Let

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\eta"] \arrow[d, "i"'] & \mathrm{path}(C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \Delta^n \arrow[r, "\sigma"'] & C \end{tikzcd} \tag{A.5.1}$$

be a commutative square in sSet. First, we will construct a lift in the case where path(C) is replaced by $\mathrm{Fun}\left(\Lambda^2_2, C\right)$. By transposing, this means that we are given a commutative square in sSet of the form

$$\begin{tikzcd} \Lambda^n_k \times \partial\Delta^1 \arrow[r, "1_{\Lambda^n_k}\times(0,1)"] \arrow[d, "i\times 1_{\partial\Delta^1}"'] & \Lambda^n_k \times \Lambda^2_2 \arrow[d, "\eta^t"] \\ \Delta^n \times \partial\Delta^1 \arrow[r, "\sigma^t"'] & C \end{tikzcd}$$

and we want to find a morphism $\ell^t\colon \Delta^n \times \Lambda^2_2 \to C$ such that both diagrams

$$\begin{tikzcd} \Lambda^n_k \times \Lambda^2_2 \arrow[r, "\eta^t"] \arrow[d, "i\times 1_{\Lambda^2_2}"'] & C \\ \Delta^n \times \Lambda^2_2 \arrow[ur, "\ell^t"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Delta^n \times \partial\Delta^1 \arrow[r, "\sigma^t"] \arrow[d, "1_{\Delta^n}\times(0,1)"'] & C \\ \Delta^n \times \Lambda^2_2 \arrow[ur, "\ell^t"'] & \end{tikzcd}$$

[1] Equality is meant to be in ho(C).

commute in sSet. Note that there is a canonical commutative square in sSet of the form

$$\begin{tikzcd}
\Lambda^n_k \times \partial\Delta^1 \arrow[r, "1_{\Lambda^n_k}\times(0,1)"] \arrow[d, "i\times 1_{\partial\Delta^1}"'] & \Lambda^n_k \times \Lambda^2_2 \arrow[d, "i\times 1_{\Lambda^2_2}"] \\
\Delta^n \times \partial\Delta^1 \arrow[r, "1_{\Delta^n}\times(0,1)"'] & \Delta^n \times \Lambda^2_2
\end{tikzcd}$$

inducing a canonical morphism $\Delta^n \times \partial\Delta^1 \cup \Lambda^n_k \times \Lambda^2_2 \to \Delta^n \times \Lambda^2_2$, which is an inner anodyne extension[1] since i is an inner anodyne extension and $(0,1)$ is a monomorphism. This reduces the problem to finding a lift of the form

$$\begin{tikzcd}
\Delta^n \times \partial\Delta^1 \cup \Lambda^n_k \times \Lambda^2_2 \arrow[r, "(\sigma^t,\eta^t)"] \arrow[d] & C \\
\Delta^n \times \Lambda^2_2. \arrow[ur, dashed, "\ell^t"'] &
\end{tikzcd}$$

Since C is quasi-category, such a lift exists.[2] Therefore, there is a commutative diagram in sSet of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\eta"] \arrow[d, "i"'] & \mathrm{Fun}\left(\Lambda^2_2, C\right) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\Delta^n \arrow[r, "\sigma"'] \arrow[ur, "\ell"] & C
\end{tikzcd}$$

Because of ℓ coincides with η at each vertex, i.e. $\ell(m) = \eta(m)$ for all natural numbers m from 0 to n, this lift restricts to a lift of (A.5.1).

$(\mathrm{ev}_0, \mathrm{ev}_1)$ is an isofibration: Let c be an object in $\mathrm{path}(C)$, i.e. a diagram in C of the form[3]

$$c_0 \xrightarrow{i_0} c_{01} \xleftarrow{i_1} c_1,$$

with i_0 and i_1 trivial cofibrations, and let $g\colon (\mathrm{ev}_0, \mathrm{ev}_1)(c) \to d$ be an isomorphism in $C \times C$. Note that $g_0\colon c \to d_0$ and $g_1\colon c \to d_1$ are isomorphisms in C, thus trivial cofibrations, and the same holds true for their respective inverses. So, $g_i^{-1} i_i\colon c_i \to d_i$ is a trivial cofibration for each $i \in \{0,1\}$ and the span diagram in C

$$d_0 \xrightarrow{i_0 g_0^{-1}} c_{01} \xleftarrow{i_1 g_1^{-1}} d_1$$

defines an object in $\mathrm{path}(C)$. Hence, the following diagram in C

$$\begin{tikzcd}
c_0 \arrow[r, "i_0"] \arrow[d, "g_0"] & c_{01} \arrow[d, "1_{c_{01}}"] & c_1 \arrow[l, "i_1"'] \arrow[d, "g_1"] \\
d_0 \arrow[r, "i_0 g_0^{-1}"'] & c_{01} & d_1 \arrow[l, "i_1 g_1^{-1}"]
\end{tikzcd}$$

commutes and defines an isomorphism in $\mathrm{path}(C)$ that $(\mathrm{ev}_0, \mathrm{ev}_1)$ sends to g.

[1] See [Cis19, Corollary 3.2.4].

[2] See [Cis19, Remark 3.3.8].

[3] In contrast to previous proofs, we try to use variable names with single letter stems and single character diacritics as often as possible in order to increase readibility.

$(\mathrm{ev}_0, \mathrm{ev}_1)$ lifts factorizations: The argument is the same as in the proof of the factorization axiom[1] for $\mathrm{path}(C)$ with the difference that the choice of factorization for f_i is already given.

$(\mathrm{ev}_0, \mathrm{ev}_1)$ lifts pseudo-factorizations: Let $f\colon a \to b$ be a map in $\mathrm{path}(C)$, i.e. a commutative diagram in C of the form

$$\begin{tikzcd} a_0 \arrow[r, "i_0"] \arrow[d, "f_0"] & a_{01} \arrow[d, "f_{01}"] & a_1 \arrow[l, "i_1"'] \arrow[d, "f_1"] \\ b_0 \arrow[r, "j_0"'] & b_{01} & b_1, \arrow[l, "j_1"] \end{tikzcd}$$

with i_0, i_1, j_0 and j_1 trivial cofibrations, and consider a pseudo-factorization of $(\mathrm{ev}_0, \mathrm{ev}_1)(f)$, i.e. commutative diagrams in C of the form

$$\begin{tikzcd} a_i \arrow[r, "f_i"] \arrow[d, "p_i"'] & b_i \arrow[d, "t_i"] \\ d_i \arrow[r, "s_i"'] & c_i \end{tikzcd}$$

with p_i a cofibration, s_i a weak equivalence and t_i a trivial cofibration for each $i \in \{0, 1\}$. Choose pushout squares in C of the form

$$\begin{tikzcd} a_i \arrow[r, "i_i"] \arrow[d, "p_i"'] & a_{01} \arrow[d, "p'_i"] \\ d_i \arrow[r, "i'_i"'] & x_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b_i \arrow[r, "j_i"] \arrow[d, "t_i"'] & b_{01} \arrow[d, "t'_i"] \\ c_i \arrow[r, "j'_i"'] & y_i \end{tikzcd}$$

with p'_i a cofibration and i'_i, j'_i and t'_i trivial cofibrations for each $i \in \{0, 1\}$. Note that the diagram in C

$$\begin{tikzcd} a_i \arrow[r, "i_i"] \arrow[d, "p_i"'] & a_{01} \arrow[d, "t'_i f_{01}"] \\ d_i \arrow[r, "j'_i s_i"'] & y_i \end{tikzcd}$$

commutes as we have in formulae:

$$j'_i s_i p_i = j'_i t_i f_i = t'_i j_i f_i = t'_i f_{01} i_i.$$

This induces a map $h_i\colon x_i \to y_i$ such that

$$\begin{tikzcd} d_i \arrow[r, "i'_i"] \arrow[d, "s_i"'] & x_i \arrow[d, "h_i"] \\ c_i \arrow[r, "j'_i"'] & y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a_{01} \arrow[r, "p'_i"] \arrow[d, "f_{01}"'] & x_i \arrow[d, "h_i"] \\ b_{01} \arrow[r, "t'_i"'] & y_i \end{tikzcd}$$

commutes for each $i \in \{0, 1\}$. By the commutativity of the left square, h_i is a weak equivalence by the 2-out-of-3 property. Again, choose pushout squares in

[1] See the last paragraph of Lemma A.2.

C of the form

$$\begin{tikzcd} a_{01} \arrow[r, "p_1'"] \arrow[d, "p_0'"'] & x_1 \arrow[d, "p_1''"] \\ x_0 \arrow[r, "p_0''"'] & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b_{01} \arrow[r, "t_1'"] \arrow[d, "t_0'"'] & y_1 \arrow[d, "t_1''"] \\ y_0 \arrow[r, "t_0''"'] & y \end{tikzcd}$$

with p_0'' and p_1'' cofibrations and t_0'' and t_1'' trivial cofibrations. (Note the unfortunate, but unavoidable inconsistency of indexing.) The diagram in C

$$\begin{tikzcd} a_{01} \arrow[r, "p_1'"] \arrow[d, "p_0'"'] & x_1 \arrow[d, "t_1''h_1"] \\ x_0 \arrow[r, "t_0''h_0"'] & y \end{tikzcd}$$

commutes as we have in formulae:

$$t_0''h_0p_0' = t_0''t_0'f_{01} = t_1''t_1'f_{01} = t_1''h_1p_1'.$$

This induces a map $h\colon x \to y$ such that

$$\begin{tikzcd} x_0 \arrow[r, "p_0''"] \arrow[d, "h_0"'] & x \arrow[d, "h"] \\ y_0 \arrow[r, "t_0''"'] & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x_1 \arrow[r, "p_1''"] \arrow[d, "h_1"'] & x \arrow[d, "h"] \\ y_1 \arrow[r, "t_1''"'] & y \end{tikzcd}$$

commutes. Now, choose a pseudo-factorization of h of the form

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "q"'] & y \arrow[d, "r"] \\ d_{01} \arrow[r, "s_{01}"'] & c_{01} \end{tikzcd}$$

with q a cofibration, s_{01} a weak equivalence and r a trivial cofibration. This concludes the constructional part of this paragraph and we are left to show that these objects form a pseudo-factorization.

Choosing composites $k_i = rt_i''j_i'$ and $\ell_i = qp_i''i_i'$ gives us span diagrams c and, respectively, d in C:

$$c_0 \xrightarrow{k_0} c_{01} \xleftarrow{k_1} c_1 \quad \text{and} \quad d_0 \xrightarrow{\ell_0} d_{01} \xleftarrow{\ell_1} d_1.$$

Because k_i is the composition of trivial cofibrations, it is a trivial cofibration, so that c is an object in path(C). As for d, we have that the diagram in C

$$\begin{tikzcd} x_i \arrow[r, "qp_i''"] \arrow[d, "h_i"'] & d_{01} \arrow[d, "s_{01}"] \\ y_i \arrow[r, "rt_i''"'] & c_{01} \end{tikzcd}$$

commutes due to the following computation in formulae:

$$rt_i''h_i = rhp_i'' = s_{01}qp_i''.$$

By the 2-out-of-3 property, qp''_i and so ℓ_i are a trivial cofibrations, so that d is an object in path(C) as well.

Furthermore, choose some composite $p_{01} = qp''_i p'_i$ for either $i \in \{0,1\}$. Because of $p''_0 p'_0 = p''_1 p'_1$, the definition of p_{01} does not depend on the choice of i. Since

$$\ell_i p_i = qp''_i i'_i p_i = qp''_i p'_i i_i = p_{01} i_i$$

holds, this diagram in C

$$\begin{tikzcd} a_0 \arrow[r, "i'_0"] \arrow[d, "p_0"] & a_{01} \arrow[d, "p_{01}"] & a_1 \arrow[l, "i'_1"'] \arrow[d, "p_1"] \\ d_0 \arrow[r, "\ell_0"'] & d_{01} & d_1 \arrow[l, "\ell_1"] \end{tikzcd}$$

commutes and thus defines a map $p\colon a \to d$ in path(C). By assumption, the component map $p_i\colon a_i \to d_i$ are cofibrations for each $i \in \{0,1\}$. Additionally, the induced map $d_i \sqcup_{a_i} a_{01} \to d_{01}$ is a cofibration because it is a composition of a canonical isomorphism $d_i \sqcup_{a_i} a_{01} \to x_i$ and qp''_i, itself a composition of two cofibrations. Hence, p is a cofibration in path(C).

By the computation in formulae

$$k_i s_i = rt''_i j'_i s_i = rt''_i h_i i'_i = rhp''_i i'_i = s_{01} qp''_i i'_i = s_{01}\ell_i,$$

this diagram in C

$$\begin{tikzcd} d_0 \arrow[r, "\ell_0"] \arrow[d, "s_0"] & d_{01} \arrow[d, "s_{01}"] & d_1 \arrow[l, "\ell_1"'] \arrow[d, "s_1"] \\ c_0 \arrow[r, "k_0"'] & c_{01} & c_1 \arrow[l, "k_1"] \end{tikzcd}$$

commutes and defines a weak equivalence $s\colon d \to c$ in path(C).

Because of $t''_0 t'_0 = t''_1 t'_1$, we may set $t_{01} = rt''_i t'_i$ for either $i \in \{0,1\}$ and note that t_{01} is a trivial cofibration in C, as a composition of such maps. Due to

$$k_i t_i = rt''_i j'_i t_i = rt''_i t'_i j_i = rt_{01}$$

this diagram in C

$$\begin{tikzcd} b_0 \arrow[r, "j_0"] \arrow[d, "t_0"] & b_{01} \arrow[d, "t_{01}"] & b_1 \arrow[l, "j_1"'] \arrow[d, "t_1"] \\ c_0 \arrow[r, "k_0"'] & c_{01} & c_1 \arrow[l, "k_1"] \end{tikzcd}$$

commutes and defines a map $t\colon b \to c$ in path(C), which is clearly a weak equivalence. Per assumption, the component maps t_0 and t_1 are cofibrations. Additionally, the induced map $c_i \sqcup_{b_i} b_{01}$ is a cofibration, as a composition of a canonical isomorphism and rt''_i, a composition of cofibrations. Hence, t is a trivial cofibration in path(C).

Finally, we show that this diagram in path(C)

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "p"'] & b \arrow[d, "t"] \\ d \arrow[r, "s"'] & c \end{tikzcd}$$

commutes by showing that it does so at each component. For $i \in \{0,1\}$, we have this diagram in C

$$\begin{tikzcd} a_i \arrow[r, "f_i"] \arrow[d, "p_i"'] & b_i \arrow[d, "t_i"] \\ d_i \arrow[r, "s_i"'] & c_i, \end{tikzcd}$$

which commutes per assumption. At the component 01, we get this diagram in C

$$\begin{tikzcd} a_{01} \arrow[r, "f_{01}"] \arrow[d, "p_{01}"'] & b_{01} \arrow[d, "t_{01}"] \\ d_{01} \arrow[r, "s_{01}"'] & c_{01}, \end{tikzcd}$$

which commutes by the computation:

$$s_{01}p_{01} = s_{01}qp_i''p_i' = rhp_i''p_i' = rt_i''h_ip_i' = rt_i''t_i'f_{01} = t_{01}f_{01}.$$

This concludes the proof that $(\mathrm{ev}_0, \mathrm{ev}_1)$ lifts pseudo-factorizations.

$(\mathrm{ev}_0, \mathrm{ev}_1)$ reflects cofibrant objects: We have shown in Lemma A.2 that all objects in $\mathrm{path}(C)$ are cofibrant. Thus, any functor out of $\mathrm{path}(C)$ reflects cofibrant objects.

$(\mathrm{ev}_0, \mathrm{ev}_1)$ reflects the 2-out-of-6 property for weak equivalences between cofibrant objects: This follows from the fact that weak equivalences in $\mathrm{path}(C)$ and $C \times C$ are defined componentwise and that the class of weak equivalences in C satisfies the 2-out-of-6 property. □

Corollary A.6. *[Szu14, Theorem 1.14]* $\mathrm{path}(C)$ *forms together with the right exact functors* $\Delta_{\Lambda^2_2} \colon C \to \mathrm{path}(C)$ *and* $(\mathrm{ev}_0, \mathrm{ev}_1) \colon \mathrm{path}(C) \to C \times C$ *a path object of* C *in* CofQCat.

Proof. By the preceding two propositions, it remains to show that the diagonal of C is a composite of $\Delta_{\Lambda^2_2}$ and $(\mathrm{ev}_0, \mathrm{ev}_1)$, which should be clear. □

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Erklärung

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in § 26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

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